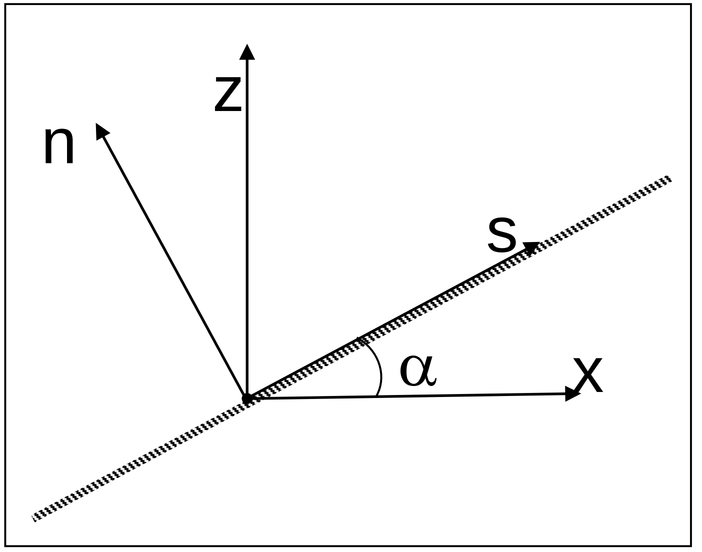
Dino Zardi
University of Trento, Italy

Stefano Serafin
University of Vienna, Austria

An analytic solution for periodic thermally driven flows over an infinite slope: Defant (1949) revisited

Prandtl (1942) - Steady thermally driven slope flow



Unperturbed basic state:

$$\theta_0(z) = \theta_{00} + \gamma z$$

$$\beta = \frac{1}{\theta_{00}}, \quad N^2 = \beta g \gamma$$

Prandtl (1942) - Governing equations

MOMENTUM:

$$0 = g\beta\theta \sin\alpha + \nu \frac{\partial^2 u}{\partial n^2}$$

ENERGY:

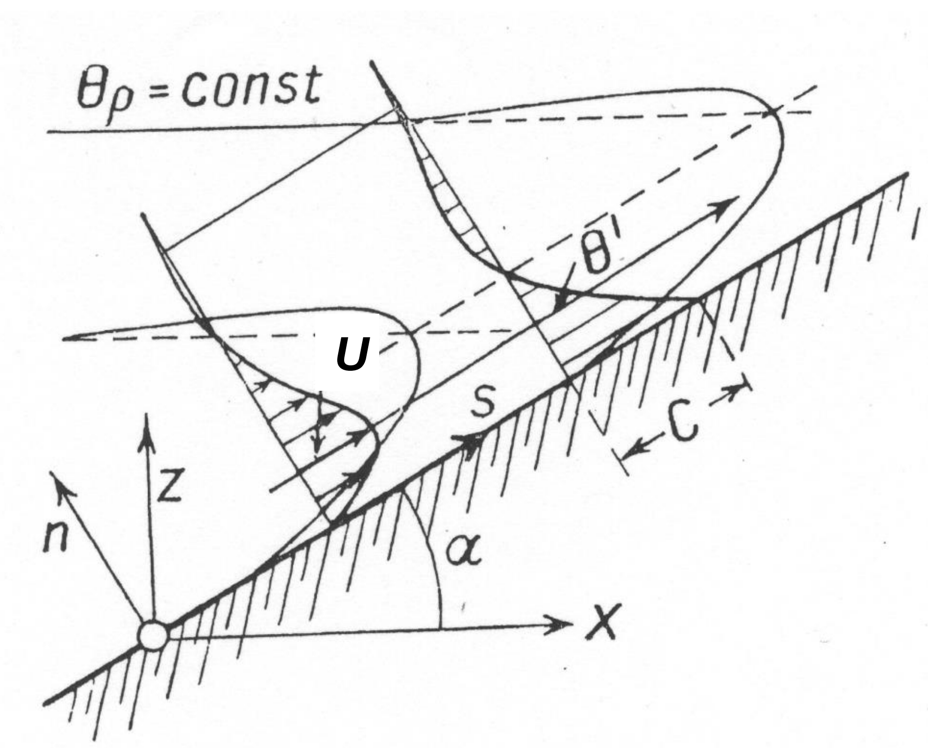
$$u\gamma \sin\alpha = \kappa \frac{\partial^2 \theta}{\partial n^2}$$

Boundary conditions:

$$u(n = 0) = 0$$

$$\theta(n = 0) = C$$

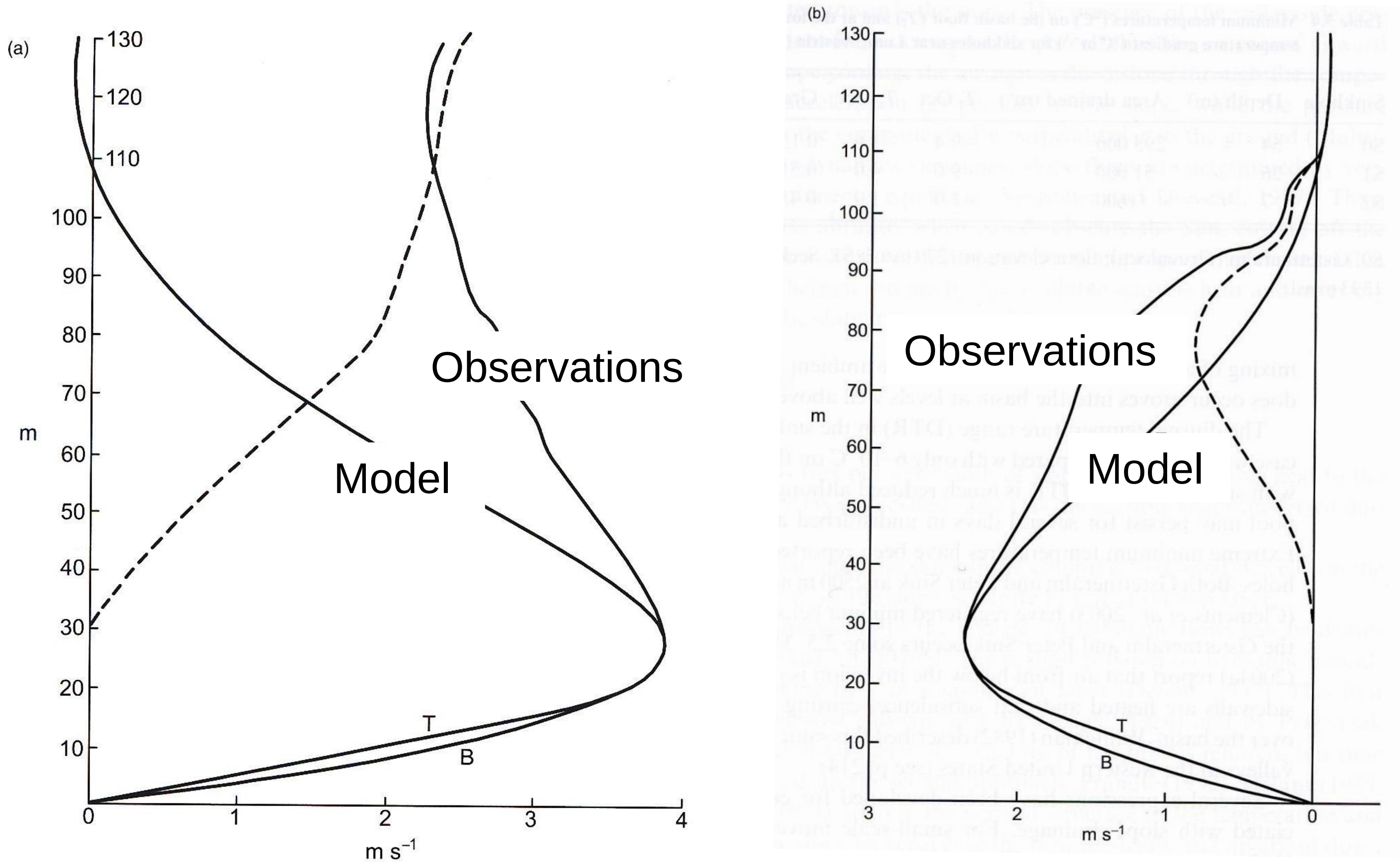
Prandtl (1942) solution



$$\begin{cases} u = U \exp\left(-\frac{n}{l}\right) \sin \frac{n}{l} \\ \theta = C \exp\left(-\frac{n}{l}\right) \cos \frac{n}{l} \end{cases}$$

$$\ell = \sqrt[4]{\frac{4\kappa\nu}{N^2 \sin^2 \alpha}}$$

$$U = \frac{N}{\gamma} \sqrt{\frac{\kappa}{\nu}} C$$



Comparison of Prandtl’s (1942) theoretical (T) slope wind profiles and balloon (B) observations of slope flows on the Nordkette near Innsbruck for (a) daytime and (b) nighttime.

Differences between theory and observations above the jet maximum are indicated by the dashed lines. (From Defant 1949, as published by Barry 2008)

Steady daily-periodic forcing: Defant (1949)

$$\frac{\partial u}{\partial t} = g\beta\theta \sin\alpha + \nu \frac{\partial^2 u}{\partial n^2}$$

$$\frac{\partial \theta}{\partial t} = -u\gamma \sin\alpha + \kappa \frac{\partial^2 \theta}{\partial n^2}$$

$$u(0,t) = 0$$

$$\theta(0,t) = C \cos \sigma t$$

Defant (1949)

Die Lösungen lauten daher vollständig

$$\vartheta = C \cdot e^{\frac{i\,n}{l}} \cdot e^{-\frac{n}{l}} \cdot e^{i\,\sigma\,t},$$

$$w = -i\,C \cdot \sqrt{\frac{a\,g\,\beta}{\nu\,B}} \cdot e^{\frac{i\,n}{l}} \cdot e^{\frac{n}{l}} \cdot e^{i\,\sigma\,t}$$

und deren reelle Teile allein

$$\vartheta = C \cdot e^{-\frac{n}{l}} \cdot \cos\frac{n}{l} \cdot \cos\sigma\,t = \vartheta_{\text{stat}} \cdot \cos\sigma\,t,$$

$$w = C \cdot \sqrt{\frac{a\,g\,\beta}{\nu\,B}} \cdot e^{-\frac{n}{l}} \cdot \sin\frac{n}{l} \cdot \cos\sigma\,t = w_{\text{stat}} \cdot \cos\sigma\,t.$$

$$\Re\left\{e^{i\frac{n}{l}}e^{i\sigma t}\right\} = \Re\left\{e^{i\left(\frac{n}{l}+\sigma t\right)}\right\} = \cos\left(\frac{n}{l}+\sigma t\right) = \cos\frac{n}{l}\cos\sigma t - \sin\frac{n}{l}\sin\sigma t$$

Reformulation of the problem
(Zardi and Serafin 2014)

Governing equations:

$$\frac{\partial u}{\partial t} = g\beta\theta \sin\alpha + K_m \frac{\partial^2 u}{\partial n^2}$$

$$\frac{\partial \theta}{\partial t} = -u\gamma \sin\alpha + K_h \frac{\partial^2 \theta}{\partial n^2}$$

Turbulence:

$$K_m = K_h \equiv K$$

Initial conditions:

$$u(0,0) = \theta(0,0) = 0$$

Boundary conditions:

$$n = 0: \quad u(0,t) = 0, \ \theta(0,t) = C \sin\omega t$$

$$n \to \infty: \quad u(n,t) \to 0, \ \theta(n,t) \to 0$$

Change variables

<u>Step 1</u>

$$v = \frac{\gamma}{N} u$$

$$\psi = \theta + i v$$

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = i N_\alpha \psi + K \dfrac{\partial^2 \psi}{\partial n^2} \\ \\ \psi(0,0) = 0 + i0 \\ \psi(0,t) = C \sin \omega t + i0 \end{cases}$$

$$N_\alpha = N \sin \alpha$$

Step 2

$$\phi = \psi\, exp(-iN_\alpha t)$$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = K\,\dfrac{\partial^2 \phi}{\partial n^2} \\ \\ \phi(0,0) = 0 + i0 \\ \phi(0,t) = C\; exp(-iN_\alpha t)\, sin\,\omega t \end{cases}$$

Periodic boundary condition

$$\phi(0,t) = C\,exp(-iN_\alpha t)\,sin\,\omega t =$$

$$= \frac{C}{2}\left[cos\left((N_\alpha + \omega)t - \frac{\pi}{2}\right) + cos\left((N_\alpha - \omega)t - \frac{\pi}{2}\right) + i\,cos(N_\alpha + \omega)t + i\,cos(N_\alpha - \omega)t\right]$$

where

$$\omega_+ = N_\alpha + \omega, \qquad \omega_- = N_\alpha - \omega,$$

$$\ell_+ = \sqrt{\frac{2K}{\omega_+}}, \qquad \ell_- = \sqrt{\frac{2K}{|\omega_-|}}$$

☛ $N_\alpha - \omega$ may be either positive, or negative, or zero

Solution

$$\phi(n,t) = \frac{iC}{\sqrt{\pi}}\int_0^{\eta}\left\{\exp\left[-i\omega_- t\left(1-\frac{\eta^2}{\mu^2}\right)\right]-\exp\left[-i\omega_+ t\left(1-\frac{\eta^2}{\mu^2}\right)\right]\right\}e^{-\mu^2}d\mu$$

$$+\frac{iC}{2}\left\{\exp\left[-i\omega_+ t-(1-i)\frac{n}{\ell_+}\right]-\exp\left[-i\omega_- t-(1-i)\frac{n}{\ell_-}\right]\right\}$$

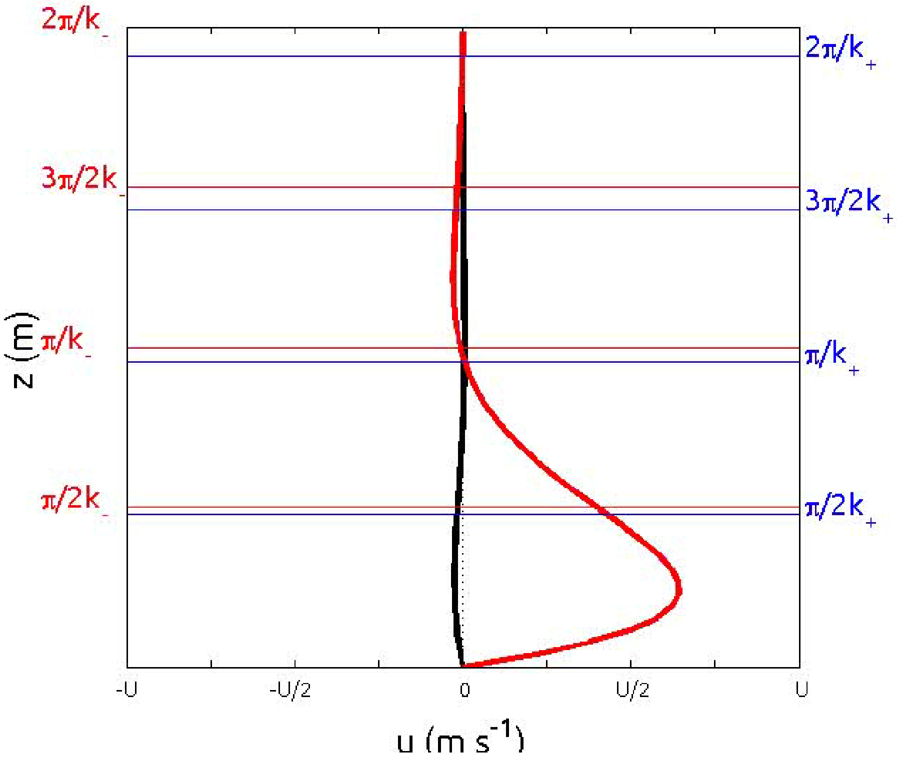
$$\eta = \frac{n}{2\sqrt{Kt}}$$

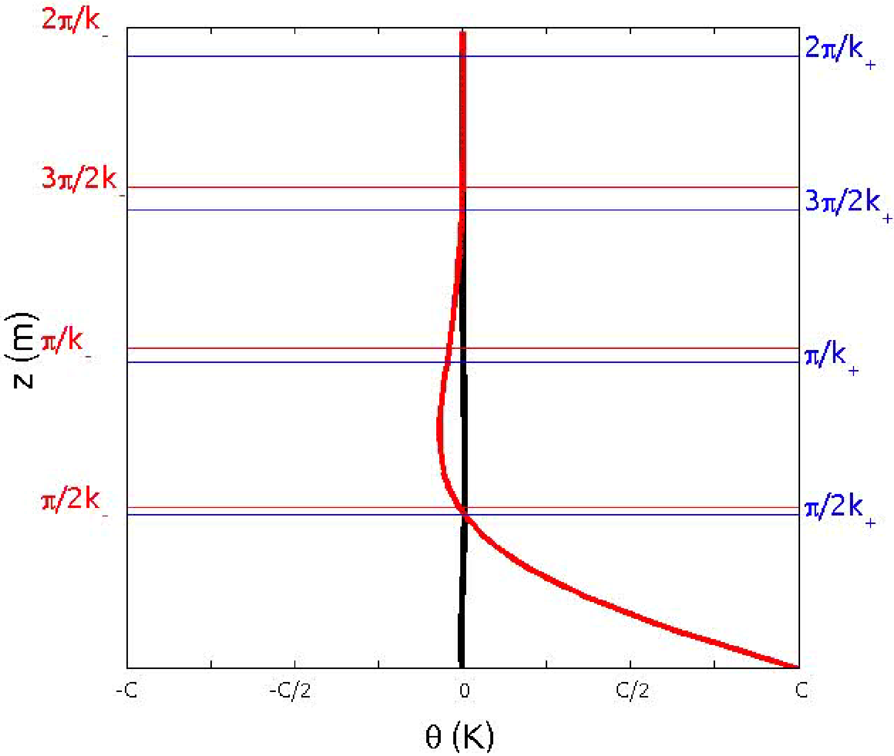
Supercritical case: $N \sin\alpha > \omega$

$$\theta(n,t) = \frac{C}{2}\left[e^{-\frac{n}{\ell_+}}\sin\left(\omega t - \frac{n}{\ell_+}\right) + e^{-\frac{n}{\ell_-}}\sin\left(\omega t + \frac{n}{\ell_-}\right)\right]$$

$$u(n,t) = \frac{N}{\gamma}\frac{C}{2}\left[e^{-\frac{n}{\ell_+}}\cos\left(\omega t - \frac{n}{\ell_+}\right) - e^{-\frac{n}{\ell_-}}\cos\left(\omega t + \frac{n}{\ell_-}\right)\right]$$

$\phi_{p'}$, $\gamma = 0.0024$ K m^{-1}, $\alpha = 10°$, $\theta_{00} = 288$ K, $\omega = 1/86400$ s^{-1}, K = 3 m^2 s^{-1}, C = 5 K, U = 9.46 ms^{-1}, $\omega t/2\pi = 0.0000$



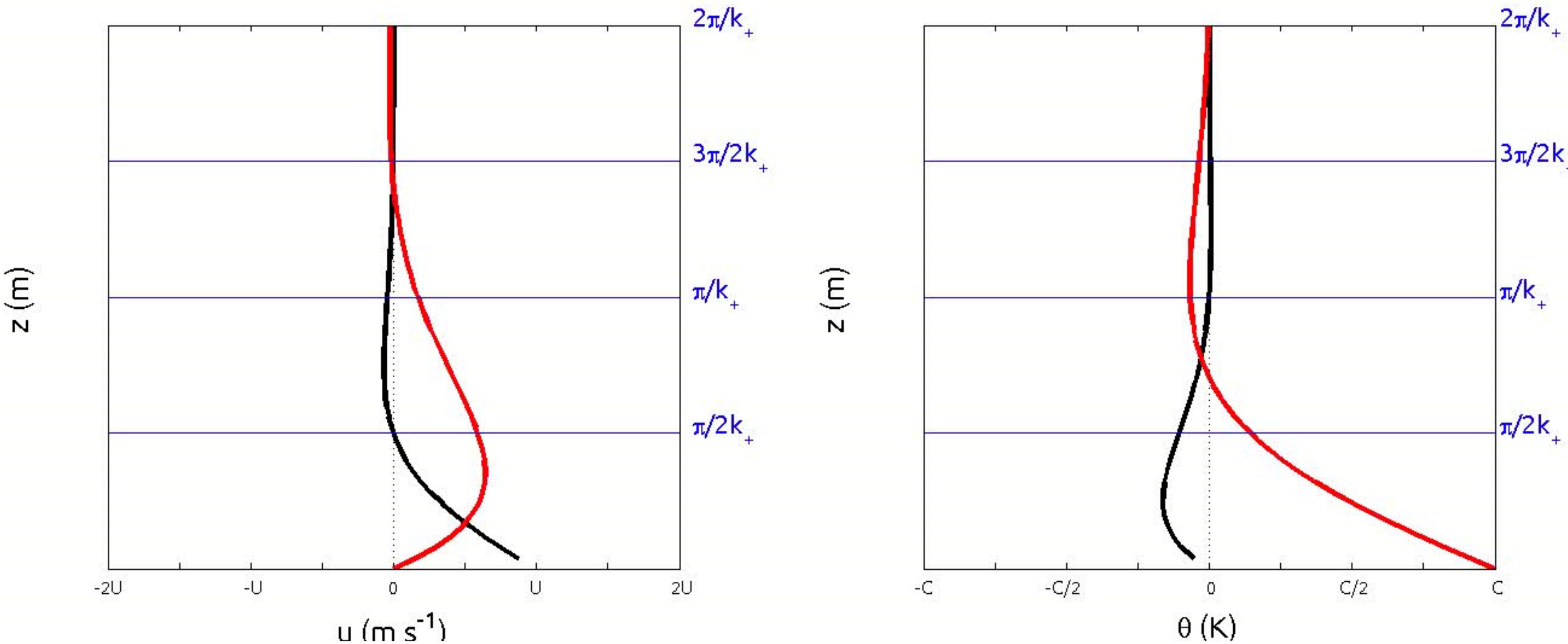


Critical case: $N \sin\alpha = \omega$

$$\theta(n,t) = \frac{C}{2}\left\{e^{-\frac{n}{\ell_+}} \sin\left(\omega t - \frac{n}{\ell_+}\right) - erfc(\eta)\sin\omega t - \frac{2}{\sqrt{\pi}}\int_0^{\eta} \sin\left[\omega t\left(1 - \frac{2\eta^2}{\mu^2}\right)\right] e^{-\mu^2} d\mu\right\}$$

$$u(n,t) = \frac{N}{\gamma}\frac{C}{2}\left\{e^{-\frac{n}{\ell_+}} \cos\left(\omega t - \frac{n}{\ell_+}\right) - erfc(\eta)\cos\omega t - \frac{2}{\sqrt{\pi}}\int_0^{\eta} \cos\left[\omega t\left(1 - \frac{2\eta^2}{\mu^2}\right)\right] e^{-\mu^2} d\mu\right\}$$

$\phi_{p'}$, $\gamma = 0.0014$ K m^{-1}, $\alpha = 0.59525°$, $\theta_{00} = 288$ K, $\omega = 1/86400$ s^{-1}, K = 3 m^2 s^{-1}, C = 5 K, U = 12.16 ms^{-1}, $\omega t/2\pi = 0.0000$

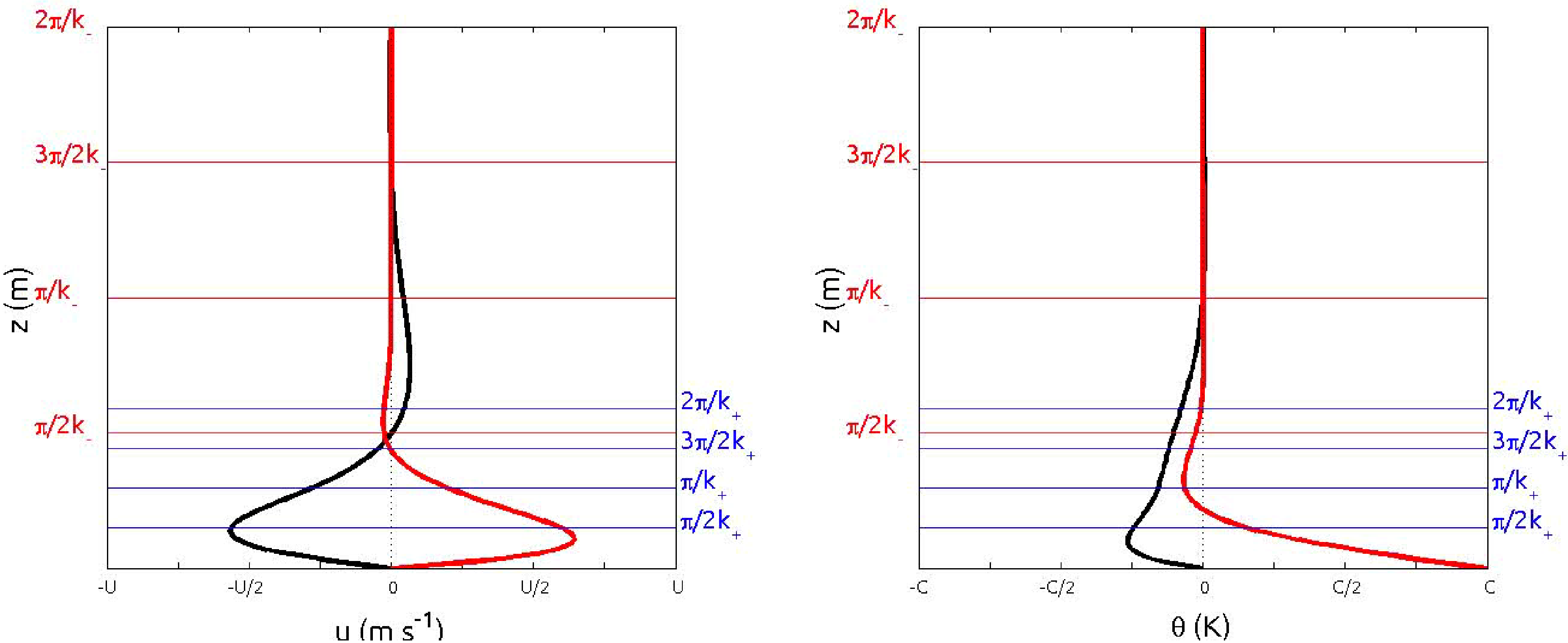


Subcritical case: $N \sin\alpha < \omega$

$$\theta(n,t) = \frac{C}{2}\left[e^{-\frac{n}{\ell_+}}\sin\left(\omega t - \frac{n}{\ell_+}\right) + e^{-\frac{n}{|\ell_-|}}\sin\left(\omega t - \frac{n}{|\ell_-|}\right)\right]$$

$$u(n,t) = \frac{N}{\gamma}\frac{C}{2}\left[e^{-\frac{n}{\ell_+}}\cos\left(\omega t - \frac{n}{\ell_+}\right) - e^{-\frac{n}{|\ell_-|}}\cos\left(\omega t - \frac{n}{|\ell_-|}\right)\right]$$

$\phi_{p'}$, $\gamma = 0.0014\ K\ m^{-1}$, $\alpha = 0.5°$, $\theta_{00} = 288\ K$, $\omega = 1/86400\ s^{-1}$, $K = 3\ m^2\ s^{-1}$, $C = 5\ K$, $U = 12.16\ ms^{-1}$, $\omega t/2\pi = 0.0000$



Conclusions

- Different regimes arise from different combinations of parameter values
- Use of a constant K provides some useful insight, but might be not realistic
- Preliminary results with variable K confirm qualitative validity of the solutions
- Stability of the solutions (especially parallel upslope flow vs. vertical convection) still to be explored

References

Defant, F., 1949: Zur Theorie der Hangwinde, nebst Bemerkungen zur Theorie der Berg- und Talwinde. [A theory of slope winds, along with remarks on the theory of mountain winds and valley winds]. *Arch. Meteor. Geophys. Bioclimatol.*, *Ser. A*, 1, 421-450 (Theoretical and Applied Climatology). [English translation: Whiteman, C.D., and E. Dreiseitl, 1984: Alpine meteorology: Translations of classic contributions by A. Wagner, E. Ekhart and F. Defant. PNL-5141 / ASCOT-84-3. Pacific Northwest Laboratory, Richland, Washington, 121 pp].

Prandtl, L., 1942: *Strömungslehre* [Flow Studies]. Vieweg und Sohn, Braunschweig, 382 pp.