RANK TWO DETECTION OF SINGULARITIES OF SCHUBERT VARIETIES

MATTHEW J. DYER

Introduction

Let G be a semisimple, simply connected complex algebraic group with Borel subgroup B, maximal torus $T \subseteq B$ and Weyl group $W = N_G(T)/T$. Identify W with the T-fixed points of the flag variety G/B by $w \mapsto \dot{w}B$ where $w = \dot{w}T$ with $\dot{w} \in N_G(T)$. For $w \in W$, the Schubert variety $X_w := \overline{B\dot{w}B/B}$ is a $l(w)$-dimensional B-stable closed subvariety of G/B with T-fixed points $\{\, v \in W \mid v \leq w \,\}$ where $\leq$ denotes Chevalley-Bruhat order on W and l is the standard length function. For background and references to the extensive literature on Schubert varieties, and in particular for the definition of rational smoothness, see [1] and also [27], [5], [31], [6]. We mention in particular that $v \leq w$ is a rationally smooth point of X_w iff there is a Zariski open neighbourhood U_v of v such that the local cohomology $H^\bullet(X_w^{\mathrm{an}}, X_w^{\mathrm{an}} \setminus \{v'\}; \mathbb{Q})$ of X_w in analytic topology is the same in each dimension as $H^\bullet(\mathbb{C}^{l(w)}, \mathbb{C}^{l(w)} \setminus \{0\}; \mathbb{Q})$, for all $v' \in U_v$. Further, each $v \leq w$ is rationally smooth in X_w iff $H^\bullet(X_w^{\mathrm{an}}; \mathbb{Q})$ satisfies Poincaré duality. A smooth point of X_w is rationally smooth. The main results of this paper entail the following criterion for smoothness and rational smoothness of T-fixed points of Schubert varieties X_w.

Theorem. (a) *For $v \leq w$ in W, v is a rationally smooth point of X_w iff for each coset Dz of each rank two (not necessarily standard) parabolic subgroup D of W such that $z \in [v, w]$ and $\sharp(Dz \cap [v, w]) \geq 3$, there is a unique $x \in Dz \cap [v, w]$ such that for each reflection t of D, either $tx < x$ or $tx \not\leq w$.*

(b) *Fix $v \in W$ with v rationally smooth in X_w. For each D, z and x as in* (a)*, let y be the minimum element of Dz in the order $\leq$ on W, and let H be a semisimple, simply connected complex algebraic group with the same root system as D and a Borel subgroup C corresponding to the roots for D which are positive for B. Then v is a smooth point of X_w iff for each Dz (and the corresponding x, y, H, C), $\dot{z}\dot{y}^{-1}C$ is a smooth point in $\overline{C\dot{x}\dot{y}^{-1}C/C}$ (a Schubert variety in H/C).*

Part (a) is a reformulation of a criterion [5] for rational smoothness in terms of the Bruhat graph (this result itself being a geometric version of Jantzen's multiplicity one criterion [24] for Verma modules in regular integral blocks of $\mathcal{O}$ for a corresponding semisimple complex Lie algebra). Note that all Schubert varieties in rank two simply laced types ($A_1 \times A_1$, A_2) are smooth, so from (b) we recover the result of Dale Peterson (see [6]) that for G simply laced of any rank (e.g. of type A,

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D or E), the rational smoothness and smoothness of a point of a Schubert variety X_w are equivalent.

Some results closely related to (a) have been obtained geometrically by [4] in the study of "pattern criteria" for smoothness and rational smoothness of Schubert varieties. It would be very interesting to have a geometric proof of the theorem in general. In this paper, the Theorem is deduced from general combinatorial results concerning the nil Hecke ring of W, using Kumar's smoothness criterion, which we now recall. Let $S = S(\mathfrak{h}^*)$ and $Q = Q(\mathfrak{h}^*)$ denote respectively the coordinate ring and the field of rational functions of $\mathfrak{h} := \mathrm{Lie}(T)$, with their W-action by $\mathbb{C}$-algebra automorphisms. Define the BGG-Demazure operators x_w for $w \in W$, regarded as $\mathbb{C}$-endomorphisms of Q, by $x_\alpha := \frac{1}{\alpha}(s_\alpha - 1_W)$ for a simple root α and $x_w := x_{\alpha_1} \cdots x_{\alpha_n}$ if $w = s_{\alpha_1} \cdots s_{\alpha_n}$, with α_i simple, is a reduced expression for w. Write $x_w = \sum_{v \in W} \hat{S}_{w,v} v$, with $\hat{S}_{w,v} \in Q$ and let $c_{w,v} = (-1)^{l(w)-l(v)} \big(\prod_{\alpha \in \Phi_+ :\, s_\alpha v \leq w} \alpha\big) \hat{S}_{w,v}$; this is a polynomial with integral coefficients in the simple roots. Kumar's smoothness criterion [31] asserts that $v \leq w$ is rationally smooth (resp., smooth) in X_w iff $c_{w,z}$ is a non-negative integer for all $v \leq z \leq w$ (resp., iff $c_{w,v} = 1$ or equivalently $c_{w,z} = 1$ for all $v \leq z \leq w$). Let us say, as an ad hoc terminology for this introduction, that v is p-smooth in X_w (where p is a prime integer) if $v \leq w$ and for each z with $v \leq z \leq w$, $c_{w,z}$ is a non-negative integer which is not divisible by p. Thus, v is smooth (resp., rationally smooth) in X_w iff it is p-smooth in X_w for all (resp., one, or equivalently, all but finitely many) prime p. Our combinatorial results imply more generally that part (b) of the above theorem holds with "p-smooth" replacing "smooth" in its last sentence.

The notion of p-smoothness arises in the study of certain characteristic p (graded) analogues of blocks of $\mathcal{O}$ which may be constructed from the reflection representation of W and (a weight poset given by) reverse Bruhat order using ideas from [21]. In fact, p-smoothness of v in X_w is equivalent to the assertion that a simple module corresponding to w appears with multiplicity one as a composition factor of a "Verma module" (universal highest weight module) corresponding to v in a certain such category, though we shall not prove this here. Using this and ideas from [34], it will be shown elsewhere that each $v \leq w$ is p-smooth in X_w iff $H^\bullet(X_w^{\mathrm{an}}; \mathbb{Z}/p\mathbb{Z})$ satisfies Poincaré duality. We do not know in general if $v \leq w$ is a p-smooth point of X_w iff there is a Zariski open neighbourhood U_v of v such that $H^\bullet(X_w^{\mathrm{an}}, X_w^{\mathrm{an}} \setminus \{v'\}; \mathbb{Z}/\mathbb{Z}p) \cong H^\bullet(\mathbb{C}^{l(w)}, \mathbb{C}^{l(w)} \setminus \{0\}; \mathbb{Z}/p\mathbb{Z})$ for all $v' \in U_v$.

The theorem remains true with semisimple algebraic groups replaced by Kac-Moody groups and rank two parabolic subgroups replaced by maximal dihedral reflection subgroups of W; the maximal dihedral subgroups are by definition the subgroups generated by the reflections in (real) roots which lie in the subspace spanned by two fixed linearly independent (real) roots. The representation-theoretic results extend to representation categories associated to reflection representations of Coxeter groups and to weight posets such as reverse Chevalley-Bruhat order, Chevalley-Bruhat order or the orders of [17] (different weight posets correspond roughly to the notion of different "blocks" of $\mathcal{O}$ for Kac-Moody Lie algebras; the categories are defined for instance over any extension field of a field obtained as the reduction modulo a maximal ideal of a number ring over which the reflection representation is defined e.g. over arbitrary fields in the case of crystallographic reflection representations). One might expect that for Weyl groups of Kac-Moody Lie algebras, the results for Chevalley-Bruhat order (resp., the orders of [17]) should also have geometric interpretations involving the (possibly infinite-dimensional) dual Schubert varieties [28] (resp., Schubert-like varieties considered in [22]).

In Section 1 of this paper, we give the general facts about the rational functions $\hat{S}_{x,w}$ on which our results depend (actually, we find it more convenient to work with $S_{x,w} := \hat{S}_{x,w}\hat{S}_{w,w}^{-1}$, where $\hat{S}_{w,w}^{-1}$ is a known product of roots). We give a family of congruences (Lemma 1.9) which characterize $S_{x,w}$ in general recursively in terms of the corresponding elements for rank two groups. In Lemma 1.13, we use this to prove an identity which, for $x \in W$ rationally smooth in X_y, gives rise recursively to expressions of $c_{y,x}$ as a product of ratios of elements $c_{a,b}$ for rank two groups. The theorem and its generalization to Kac-Moody groups is obtained as an immediate consequence of this and Kumar's smoothness criterion at the start of Section 3. Representation-theoretically, the combinatorics in 1.1–1.17 is most naturally regarded as attached to a weight poset of reverse Chevalley-Bruhat order; we indicate in 1.19–1.21 how these results extend to other weight posets (e.g. using Chevalley-Bruhat order instead, one gets by (1.20.1) similar results about the "inverse matrix" of $(S_{y,x})_{y,x\in W}$).

The theme of Section 1 may be summarized as "reduction to rank two." In Section 2, we study the rank two situation in more detail. The main results are an explicit formula (Proposition 2.25) for $c_{w,v}$ and information on its divisibility properties in the case of dihedral groups. In the case of the affine Weyl group of type $A_1^{(1)}$, the $c_{w,v}$ are ordinary binomial coefficients; in general, they are of the form $\frac{c_n c_{n-1}\dots c_{n-k+1}}{c_k c_{k-1}\dots c_1}$ where c_i denote certain (naturally parameterized) root coefficients. Using the result 2.15 (which is proved using the analogue of an identity [32, 34.1.2(c)] on Gaussian binomial coefficients which is used in the construction of the quantum Frobenius homomorphism), we are able to give in Section 3 a complete description of the p-singular loci in Kac-Moody groups of rank two. Although we have no need for it in this paper, we also give as Proposition 2.23 the analogue of the well-known formula for the p-adic valuation of an ordinary binomial coefficient.

Section 3 is devoted to applications of the results in Sections 1–2. We give our characterization of singular loci of Schubert varieties in Kac-Moody groups, and provide some examples; in particular, we explicitly list the singular loci of Schubert varieties in rank two Kac-Moody groups since these are required ingredients in our general description. The results for rank two groups are well-known at least in the finite case; in the general rank two case, X_w is singular in codimension two unless perhaps w is of length at most 6.

Our original proof of the general facts underlying the above Theorem is in part more representation-theoretic in nature, and yields as well various stronger results that I am unable to prove by the methods of this paper; however, it requires a large (by comparison with the proofs here) amount of background information, only some of which is given in [21], [12], [13]. For this reason, we have given here largely self-contained (though possibly less perspicuous) combinatorial proofs. At the end of Section 3, we provide some indications without proof on the relations to representation theory, and we defer a more systematic discussion of these to future papers. In the case of finite Weyl groups, for which the stronger results are probably of greatest interest, they can be obtained by combining results in [4] with the combinatorial arguments of this paper, as we indicate.

I thank Jim Carrell for providing me with a copy of [6].

1. General Results

As general references for Coxeter groups, root systems and Chevalley-Bruhat order, consult [3] and [23].

1.1. We consider contragredient reflection representations of a Coxeter system (W,R) on real vector spaces V, V' endowed with a W-invariant $\mathbb{R}$-bilinear map $\langle\ ,\ \rangle\colon V\times V'\to\mathbb{R}$. Let $\Pi\subseteq\Phi_+\subseteq\Phi$ denote the simple roots, positive roots and roots in V respectively, and $\Pi^\vee\subseteq\Phi_+^\vee\subseteq\Phi^\vee\subseteq V'$ denote the simple coroots, positive coroots and coroots in V', respectively. The reflection in a root α or corresponding coroot $\alpha^\vee$ will be denoted s_α. We regard W as a subgroup of $\mathrm{GL}(V)$ unless otherwise indicated.

Our results apply to standard (real, reduced) root systems and corresponding reflection representations of reductive complex algebraic groups ([2]) and Kac-Moody groups (for which we follow conventions as in [30]) and to general Coxeter groups ([3], [23]). For a precise description of our technical assumptions on the root system and some of their implications, see Appendix A. In particular, we assume in Sections 1–3 that the simple roots are $\mathbb{R}$-linearly independent and the root system is reduced, unless otherwise indicated.

1.2. Let $l'\colon W\to\mathbb{N}$ denote the standard length function on W and $l\colon W\to\mathbb{Z}$ be defined by $l(w)=-l'(w)$. We abbreviate $l(y,x)=l(x)-l(y)$. Define the directed, edge-labeled graph $\Omega=\Omega_W$ with vertex set W and labeled edges $y\xrightarrow{\alpha}x$ for y,x in W and $\alpha\in\Phi_+$ with $l(y)<l(x)$, and $y=s_\alpha x$ (omitting the labels and reversing the arrows gives the Bruhat graph of W, as defined in [16], for instance). Define the partial order $\leq$ on W by $v\leq w$ iff there is a directed path $v=v_0\to v_1\to\cdots\to v_n=w$ in Ω; $\leq$ is reverse Chevalley-Bruhat order on W, with 1_W as maximum element (note that this differs from our use of $\leq$ in the Introduction). The following well-known "Z-property" [8] plays an important implicit role in many of our inductive proofs; for $y,x\in W$ and a simple reflection s with $sy>y$ and $sx<x$, we have

$$y\leq x\iff y\leq sx\iff sy\leq x. \tag{1.2.1}$$

For any $y\leq x$ in W set $\Phi_{y,x}:=\{\,\alpha\in\Phi_+\mid y\leq u\xrightarrow{\alpha}x\,\}$. Using the Z-property, we have that for $\alpha\in\Pi$ and $\gamma\in\Phi_+$,

$$z\xrightarrow{\gamma}w\quad\text{iff}\quad s_\alpha z\xrightarrow{s_\alpha(\gamma)}s_\alpha w,\qquad\text{provided }\gamma\neq\alpha. \tag{1.2.2}$$

1.3. Let $S(V)$ denote the symmetric algebra over $\mathbb{R}$ of V and let $Q(V)$ denote the quotient field of $S(V)$, with their natural W-actions by ring automorphisms induced by the W-action on V. Following [29], form the associative, unital $\mathbb{R}$-algebra Q_W which has elements δ_v for $v\in W$ as basis as left $Q(V)$ vector space, with $\mathbb{R}$-bilinear multiplication given by $(q\delta_v)(q'\delta_w)=qv(q')\delta_{vw}$ for $v,w\in W$ and $q,q'\in Q(V)$. There is a faithful representation $\theta\colon Q_W\to\mathrm{End}_{\mathbb{R}}(Q(V))$ with $\theta(q\delta_v)q'=qv(q')$.

For $\alpha\in\Pi$, define $x_\alpha:=\frac{1}{\alpha}\delta_{s_\alpha}-\frac{1}{\alpha}\delta_{1_W}\in Q_W$. It is known from [29, Proposition 4.2] that $x_\alpha^2=0$ and that for w in W, there is a unique element (BGG-Demazure operator) $x_w\in Q_W$ such that $x_w=x_{\alpha_1}\cdots x_{\alpha_n}$ whenever $\alpha_i\in\Pi$ are such that $w=s_{\alpha_1}\cdots s_{\alpha_n}$ is a reduced expression for w (the x_w for $w\in W$ form a left $S(V)$-module basis for a subring of Q_W which has been called the nil Hecke ring of W in loc cit).

1.4. Write $x_w = \sum_{v \in W} \hat{S}_{w,v}\delta_v$, with $\hat{S}_{w,v} \in Q(V)$ (the element $\hat{S}_{w,v}$ is denoted as $c_{w^{-1},v-1}$ in [29], as $c_{w,v}$ in [1] and as $S^{\emptyset}_{v,w}$ in [12]). The equation

$$x_\alpha x_w = \begin{cases} x_{s_\alpha w} & \text{if } l(sw) < l(w) \\ 0 & \text{otherwise} \end{cases} \tag{1.4.1}$$

for $\alpha \in \Pi$ and $s = s_\alpha$ readily gives the recurrence formulae

$$\hat{S}_{w,z} = \frac{1}{\alpha}\big(s(\hat{S}_{sw,sz}) - \hat{S}_{sw,z}\big) = \hat{S}_{w,sz}, \quad \text{if } l(sw) > l(w) \tag{1.4.2}$$

$$\hat{S}_{w,w} = \frac{1}{\alpha}s(\hat{S}_{sw,sw}) \quad \text{if } l(sw) > w \tag{1.4.3}$$

$$\hat{S}_{1,1} = 1, \qquad \hat{S}_{w,z} = 0 \quad \text{unless } w \le z. \tag{1.4.4}$$

1.5. By (1.4.3)–(1.4.4), $S_{v,v} \neq 0$ for $v \in W$. Define $S_{w,v} = \frac{\hat{S}_{w,v}}{\hat{S}_{v,v}} \in Q(V)$ for any $v, w \in W$. Write $\mathcal{P}$ for the set of closed intervals $[y, x]$ with $y \le x$ in W.

Lemma. *The elements $S_{y,x}$ defined for $y, x \in W$ are uniquely determined by the following conditions:*

(i) $S_{y,x} = 0$ *if* $y \not\le x$.
(ii) $S_{1,1} = 1$.
(iii) $S_{y,x} = -\frac{1}{\alpha}s(S_{y,sx})$ *if* $\alpha \in \Pi$, $s = s_\alpha$, $sx < x$ *and* $sy > y$.
(iv) $S_{sy,x} = -\frac{1}{\alpha}s(S_{sy,sx}) + s(S_{y,sx})$ $\alpha \in \Pi$, $s = s_\alpha$, $sx < x$ *and* $sy > y$.

Proof. The conditions follow using (1.4.2) and (1.4.3); uniqueness follows from the Z-property. □

1.6. Let S^{++} denote the set of polynomials $u \in S(V)$ such u is a $\mathbb{R}_{\ge 0}$-linear combination of products of positive roots, and let S^+ denote the set of elements f of $S(V)$ such that $fu = u'$ for some $u, u' \in S^{++}$ with $u \neq 0$. We give $S(V)$ the $\mathbb{N}$-grading with $S(V)_0 = \mathbb{R}$ and $S(V)_1 = V$.

The following Theorem is proved in [19].

Theorem. (a) *For* $[y, x] \in \mathcal{P}$, $S_{y,x} = (-1)^{l(x)-l(y)}\frac{c_{y,x}}{\prod_{\alpha \in \Phi_{y,x}} \alpha}$ *for some non-zero* $c_{y,x} \in S^+$, *with* $c_{y,x}$ *homogeneous of degree* $\sharp(\Phi_{y,x}) - l(y, x)$ *in* $S(V)$.
(b) (Deodhar's inequality) $\sharp(\Phi_{y,x}) \ge l(y, x)$ *for* $[y, x] \in \mathcal{P}$.

Part (b) was proved geometrically for symmetric groups by Deodhar [10], and for crystallographic W (resp., finite Weyl groups W) it was proved in [5] (resp., [33]). Part (a) with $c_{y,x} \in S$ in place of $c_{y,x} \in S^+$ is also proved geometrically for crystallographic W in [31].

1.7. To any reflection subgroup D of W, there is a canonically associated root subsystem of Φ with positive roots $\Phi^D_+ := \{\, \alpha \in \Phi_+ \mid s_\alpha \in D \,\}$ (the simple roots of Φ^D_+ are not necessarily linearly independent if W is infinite, see Appendix A). In applying a notion defined for general Coxeter groups to D, we always consider D as a Coxeter system with simple reflections corresponding to the simple roots of Φ^D_+ (i.e. we consider D with its canonical set of Coxeter generators in the sense of [15]).

We say that a subgroup D of W is a maximal dihedral reflection subgroup of W if D is a subgroup of W generated by all the reflections in roots lying on some fixed two-dimensional subspace U of V which is spanned by a subset of Φ (such

D necessarily have exactly two simple roots (by [16] or [14]) which are of course linearly independent). Let M (resp., $M_{(\gamma)}$) denote the set of all maximal dihedral reflection subgroups of W (resp., all those which contain a fixed $\gamma \in \Phi$). Observe that for $\gamma \in \Phi$, each reflection of W except s_γ is contained in a unique element of $M_{(\gamma)}$.

1.8. For any any edge-labeled directed graph Γ and any subset X of the vertex set of Γ, let $\Gamma(X)$ denote the full edge-labeled subgraph of Γ on vertex set X. We shall use the following fact which follows from [16]:

Lemma. *Let D be any reflection subgroup of W. Define the edge labeled Bruhat graph Ω_D of D. For any $y \in W$, let z denote the maximum element of the coset Dy in $\leq$. Then there is an isomorphism of labeled directed graphs $\Omega_W(Dy) \to \Omega_D$ induced by the map $p \mapsto py^{-1}$ of their vertex sets.*

Recall from Theorem 1.6 the elements $c_{z,w} := (-1)^{l(w)-l(z)} S_{z,w} \prod_{\alpha\in\Phi_{z,w}} \alpha$ for any $z, w \in W$. For any reflection subgroup D of W and any $z, w \in W$ with $Dz = Dw$, define $S^D_{z,w}$, $\Phi^D_{z,w}$, $c^D_{w,z}$ as follows; let p be the maximum element of Dz in the order $\leq$ on W and set $S^D_{z,w} := S_{zp^{-1},wp^{-1}}$, $\Phi^D_{z,w} := \Phi_{zp^{-1},wp^{-1}}$ and $c^D_{z,w} := c_{zp^{-1},wp^{-1}}$ where the right hand sides are computed in D (with positive roots Φ^D_+ and reverse Chevalley-Bruhat order). Similarly, for $z, w \in W$ with $Dz = Dw$, we write $l_D(z) := l''(zp^{-1})$, $l_D(z, w) := l''(wp^{-1}) - l''(zp^{-1})$, $z \leq_D w$ if $zp^{-1} \leq'' wp^{-1}$, where $-l''$ denotes the standard length function on D and $\leq''$ denotes reverse Chevalley-Bruhat order on D (which is not in general the restriction of reverse Chevalley-Bruhat order on W).

1.9. Fix for the remainder of Section 1 a subring A of $\mathbb{R}$ such that $\langle \alpha, \beta^\vee \rangle \in A$ for all $\alpha \neq \beta \in \Pi$ (equivalently, for all $\alpha \neq \beta \in \Phi$). In this case, we shall say that Φ and $\Phi^\vee$ are defined over A. We have $\Phi \subseteq A\Pi$ and similarly for $\Phi^\vee$. For example, one could take $A = \mathbb{Z}$ if the root system arises from a semisimple algebraic group or Kac-Moody group; in general, one could take $A = \mathbb{R}$, or A as a suitable number ring in the case of the standard reflection representation of W.

Let $A[\Pi]$ be the subring of $Q(V)$ generated by $A \cup \Pi$, and let B be any subring of $Q(V)$ containing $A[\Pi]$. Denote by $B_{(\emptyset)}$ (resp., $B_{(\gamma)}$ for $\gamma \in \Phi_+$) the localization of B at the multiplicative subset generated by Φ (resp., by $\Phi \setminus \{\pm\gamma\}$).

Lemma. (a) *$S_{y,x} \in B_{(\emptyset)}$ and $c_{y,x} \in B$ for all $[y, x] \in \mathcal{P}$*

(b) *If $z \xrightarrow{\gamma} w$ and $[z, w] \in \mathcal{P}$, then $S_{z,w} \equiv -\frac{1}{\gamma} \prod_{D\in M_\gamma} (-\gamma S^D_{z,w}) \pmod{B_{(\gamma)}}$.*

(c) *If $y \leq z \xrightarrow{\gamma} x$ and $[y, x] \in \mathcal{P}$, then $S_{y,x} \equiv S_{y,z} S_{z,x} \pmod{B_{(\gamma)}}$.*

Proof. For the proof, we take $B = A[\Pi]$ without loss of generality. Consider $\alpha \in \Pi$, $x, y \in W$ satisfying $sy > y$ and $sx < x$, where $s = s_\alpha$. We have that

$$\Phi_{sy,x} \subseteq \Phi_{y,x} = s(\Phi_{y,sx}) \cup \{\alpha\}, \qquad \Phi_{sy,sx} \subseteq \Phi_{y,sx}$$

Note also that if $\gamma \in \Phi_+ \setminus \alpha$ and $sy \xrightarrow{\gamma} x$ (equivalently, $y \xrightarrow{s\gamma} sx$) then $s\gamma \notin \Phi_{sy,sx}$ while if $\gamma \in \Phi_+ \setminus \alpha$ and $y \xrightarrow{\gamma} x$ (equivalently, $sy \xrightarrow{s\gamma} sx$) then $\gamma \notin \Phi_{sy,x}$. By Lemma 1.5, we get

$$S_{y,x} = -\frac{1}{\alpha} S_{sy,x} - \frac{1}{\alpha^2} s(S_{sy,sx}) \tag{1.9.1}$$

$$S_{sy,x} = -\frac{1}{\alpha} s(S_{sy,sx}) + s(S_{y,sx}). \tag{1.9.2}$$

The elements $S_{v,w}$ are uniquely determined by (1.9.1)–(1.9.2) and the initial values $S_{1,z} = \delta_{1,z}$ for $z \in W$. Using them, one sees that $S_{v,w}$ is actually in the subring of $Q(V)$ generated by (1 and) the elements $\frac{1}{\alpha}$ for $\alpha \in \Phi$. In particular, $S_{v,w} \in B_{(\emptyset)}$. Comparing with 1.6(a), we see that $c_{v,w} \in \mathbb{R}[\Pi] \cap B_{(\emptyset)} = B$ (specifically, we have $c_{v,w} \in \mathbb{R}[\Pi]$, $c_{v,w}\beta_1 \cdots \beta_n \in B = A[\Pi]$ for some $\beta_i \in \Phi$; since B (resp., $\mathbb{R}[\Pi]$) is a polynomial ring over A (resp., $\mathbb{R}$) on generators $z(\Pi)$ for any fixed $z \in W$, each β_i is part of a set of polynomial generators of $\mathbb{R}[\Pi]$ over $\mathbb{R}$ and of B over A, so obviously $c_{v,w} \in B$). Using 1.6(a) and the remarks at the start of the proof gives

$$S_{y,x} \equiv -\frac{1}{\alpha^2} s(S_{sy,sx}) \pmod{B_{(\gamma)}} \qquad \text{if } y \xrightarrow{\gamma} x,\ \gamma \neq \alpha \tag{1.9.3}$$

$$S_{sy,x} \equiv s(S_{y,sx}) \pmod{B_{(\gamma)}} \qquad \text{if } sy \xrightarrow{\gamma} x,\ \gamma \neq \alpha \tag{1.9.4}$$

$$S_{z,w} = -\frac{1}{\gamma} \qquad \text{if } z \xrightarrow{\gamma} w \text{ and } l(w) - l(z) = 1. \tag{1.9.5}$$

It is clear that the elements $S_{z,w}$ with $z \xrightarrow{\gamma} w$ for some $\gamma \in \Phi_+$ are uniquely determined modulo $B_{(\gamma)}$ by the recurrence equations (1.9.3)–(1.9.4) and initial conditions (1.9.5).

Let $S'_{z,w} := -\frac{1}{\gamma} \prod_{D \in M_\gamma} (-\gamma S^D_{z,w})$ for any $z \xrightarrow{\gamma} w$ and any $\gamma \in \Phi_+$ (although this is possibly an infinite product, all but finitely many of its factors are equal to 1 by (1.9.5) and the general fact (1.11.2) below). There is exactly one $D_s \in M_\gamma$ with $s_\alpha \in D_s$. One has $sD_s s = D_s \in M_\gamma \cap M_{s(\gamma)}$. Further, the equations above hold with each $S_{z,w}$ replaced by $S^{D_s}_{z,w}$, using Lemma1.8. For $D \in M_\gamma$ with $D \neq D_s$, we have $sDs^{-1} \in M_{s(\gamma)}$ and $S^D_{z,w} = s(S^{sDs}_{sz,sw})$ for any $z, w \in W$ with $Dz = Dw$ (note $s(\Phi^D_+) = \Phi_+(sDs)$). Since the map $D \mapsto sDs$ gives a bijection $M_\gamma \to M_{s(\gamma)}$, one can now check that (1.9.3)–(1.9.5) also hold modulo $B_{(\gamma)}$ with each $S_{a,b}$ replaced by $S'_{a,b}$. This proves (b).

The proof of (c) is similar, using both sets of recurrence relations above. Observe first that if $z \xrightarrow{\gamma} x$, then for any $y \in W$, $S_{y,z}S_{z,x} \equiv 0 \equiv S_{y,x} \pmod{(B_{(\gamma)})}$ if $y \not\leq z$, by Theorem 1.6(a). Fix again for the remainder of the proof $\alpha \in \Pi$, $x, y \in W$ satisfying $sy > y$ and $sx < x$, where $s = s_\alpha$. We next check (c) in case $\gamma = \alpha \in \Pi$. If $sy \leq sx \xrightarrow{\alpha} x$, we have $S_{sy,x} \equiv -\frac{1}{\alpha} s(S_{sy,sx}) \equiv S_{sy,sx}S_{sx,x} \pmod{B_{(\alpha)}}$ by Lemma 1.5(iv) and Theorem 1.6(a), since $s(S_{y,sx}) \equiv S_{y,sx} \pmod{\alpha B_{(\alpha)}}$. Similarly, we have $S_{y,x} = -\frac{1}{\alpha} s(S_{y,sx}) \equiv S_{y,sx}S_{sx,x} \pmod{B_{(\alpha)}}$. Now fix $\gamma \neq \alpha$ with $y \leq z \xrightarrow{\gamma} x$. To prove (c) by induction, it will suffice by the Z-property to verify that if $S_{sy,sx} \equiv S_{sy,sx}S_{sz,sx} \pmod{B_{(\gamma)}}$, then the following two claims hold:

(i) $S_{sy,x} \equiv S_{sy,z}S_{z,x} \pmod{B_{(\gamma)}}$ iff $S_{y,sx} \equiv S_{y,sz}S_{sz,sx} \pmod{B_{(s\gamma)}}$
(ii) if the equivalent conditions of (i) hold, then $S_{y,x} \equiv S_{y,z}S_{z,x} \pmod{B_{(\gamma)}}$.

Assume the second condition in (i) holds. Then repeatedly using Theorem 1.6(a),

$$\begin{aligned} S_{sy,x} &\equiv -\frac{1}{\alpha} s(S_{sy,sz}S_{sz,sx}) + s(S_{y,sz}S_{sz,sx}) \\ &\equiv \Big(-\frac{1}{\alpha} s(S_{sy,sz}) + s(S_{y,sz})\Big) \cdot \begin{cases} S_{z,x} & \text{if } sz < z \\ -\alpha^2 S_{z,x} & \text{if } sz > z \end{cases} = S_{sy,z}S_{z,x} \end{aligned}$$

where the congruences are taken modulo $B_{(\gamma)}$. Similarly, the first condition in (i) implies the second. To prove (ii), assume that both conditions in (i) hold and compute again modulo $B_{(\gamma)}$ that

$$\begin{aligned} S_{y,x} &\equiv -\frac{1}{\alpha} S_{sy,z} S_{z,x} - \frac{1}{\alpha^2} s(S_{sy,sz} S_{sz,sx}) \\ &\equiv S_{z,x} \Big(-\frac{1}{\alpha} S_{sy,z} + s(S_{sy,sz}) \cdot \begin{cases} -\frac{1}{\alpha^2} & \text{if } sz < z \\ 1 & \text{if } sz > z \end{cases} \Big) \equiv S_{y,z} S_{z,x} \end{aligned}$$

□

Remarks. Let $[y, x] \in \mathcal{P}$. If $z \xrightarrow{\gamma} x$, then the congruences above can be combined to give $S_{y,x} \equiv -\frac{1}{\gamma} S_{y,z} \prod_{D \in M_\gamma} (-\gamma S^D_{z,w}) \pmod{B_{(\gamma)}}$. Taking $B = A[\Pi]$, we have $\cap_{\gamma \in \Phi_+} B_{(\gamma)} \subseteq B$, and it follows that $S_{y,x}$ is completely determined (recursively) for $y \le x$ by the values of the $S^D_{z,w}$ (using these congruences for $y < x$, the initial condition that $S_{x,x} = 1$ and the condition that $S_{y,x}$ is homogeneous of degree $-(l(x) - l(y))$, using the grading of $B_\emptyset$ with $A \subseteq (B_\emptyset)_0$ and $\Pi \subseteq (B_\emptyset)_1$). To use these facts to characterize the elements $S_{y,x}$, it is therefore enough to describe the $S_{z,w}$ in dihedral groups D; this we shall do in Section 2.

1.10. For any interval $[y, x] \in \mathcal{P}$ and any coset Dz of a maximal dihedral subgroup D, Lemma 1.8 implies that $\Omega(Dz \cap [y, x])$ is isomorphic to a full subgraph of Ω_D with vertex set equal to a (open, closed or half-open) interval X in reverse Chevalley-Bruhat order on D. We briefly discuss the structure of these graphs.

Consider $G = \mathbb{Z}$ as a poset with the partial order $\preceq$ such that $a \prec b$ iff $|a| < |b|$. Give G the structure of a directed graph with an edge $a \to b$ if $a \preceq b$ and $b - a$ is odd (there are no loops, multiple edges or directed cycles in G). See Figure 1.

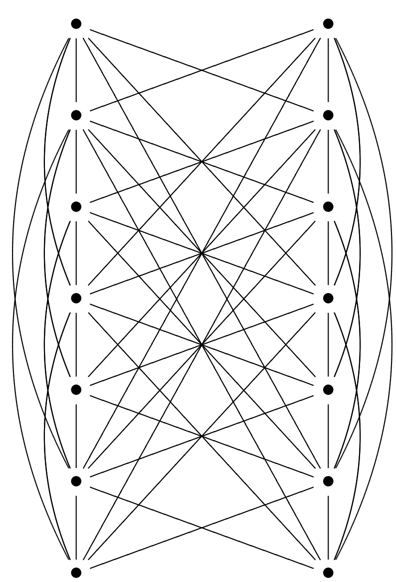


FIGURE 1. Diagram of a full subgraph of G with vertex set given by an open interval; all edges are directed upwards, and we have omitted arrows for clarity.

If D is a dihedral group, then every open, closed or half-open interval X in reverse Chevalley-Bruhat order $\leq_D$ on D is isomorphic to an interval X' of corresponding type in the order $\preceq$ on G, and then $\Omega_D(X) \cong G(X')$ as directed graph. We observe that for such X, the underlying undirected graph of $\Omega_D(X)$ is connected and has more than one vertex iff it contains an edge; in that case, the underlying undirected graph is a complete bipartite graph. Otherwise, $\Omega_D(X)$ either consists of one or two isolated vertices (and no edges), or has no vertices.

We say that a vertex w of a directed graph Γ is a source (resp., sink) if there is a directed path from (resp., to) w to (resp., from) each vertex t of Γ. In any finite directed graph without loops or directed cycles, a source (resp., sink) is unique if it exists. It is clear that for X as above, $\Omega_D(X)$ has a source (resp., sink) iff X has a minimum (resp., maximum) element in the order $\leq_D$. Moreover, if $X \neq \emptyset$ is closed (i.e. has both a minimum element m and a maximum element M) then every vertex of $\Omega_D(X)$ is joined by an (undirected) edge to exactly $n := l_D(M) - l_D(m)$ vertices of $\Omega_D(X)$, where $-l_D$ denotes the standard length function of D, and further, any totally ordered subset of X is contained in a totally ordered subset of X of cardinality $n+1$.

1.11. The condition (ii) in the following theorem first appeared in Jantzen's multiplicity one criterion for Verma modules [24].

Theorem. *For $[y,x] \in \mathcal{P}$, the following conditions are equivalent:*

(i) $c_{y,z} \in \mathbb{R}$ *for all* $z \in [y,x]$
(ii) *for each* $z \in [y,x]$, $\sharp(\Phi_{y,z}) = l(y,z)$.
(iii) *for each* $D \in M$ *and* $z \in [y,x]$ *such that* $\Omega(Dz \cap [y,x])$ *has an edge, it has a source.*

We shall say that the interval $[y,x]$ is $\mathbb{R}$-smooth if it satisfies these equivalent conditions (this property depends only on the order type of the interval $[y,x]$ by [16]).

Remarks. (1) If $\Omega(Dz \cap [y,x])$ has a source, this source is necessarily a minimum element of $Dz \cap [y,x]$ in the order $\leq$ on W; however, it is easy to give examples to show that in general $Dz \cap [y,x]$ may have a minimum element in $\leq$ which is not a source of $\Omega(Dz \cap [y,x])$, even if the latter graph has an edge.

(2) The hypothesis in (iii) that $\Omega(Dz \cap [y,x])$ has an edge could be replaced by any one of the (inequivalent) hypotheses that $\sharp(Dz \cap [y,x]) \geq 3$, or that (the undirected graph underlying) $\Omega(Dz \cap [y,x])$ is connected, or that $\Omega(Dz \cap [y,x])$ has at least two edges, or that $\Omega(Dz \cap [y,x])$ does not consist of two isolated vertices (with no edges). The most natural formulation of condition (iii) is perhaps that for each $D \in M$ and $z \in [y,x]$, there is a unique vertex z' of $\Gamma := \Omega(Dz \cap [y,x])$ such that there is a directed path $z' \to \cdots \to z$ (with $n \geq 0$ edges) in Γ but no edge $z'' \to z'$ in Γ.

Proof. The equivalence of (i) and (ii) is obvious from Theorem 1.6. We shall prove the equivalence of (ii) and (iii) by induction on $n := l(y,x)$. For $n \leq 1$, (ii) and (iii) both hold. Suppose inductively that the equivalence is true for intervals of length less than n. Let $[y,x]$ be such that $l(y,x) = n$ and for each z with $y \leq z < x$, $\sharp(\Phi_{y,z}) = l(y,z)$ (equivalently by the inductive hypothesis, for each such z, $\Omega(Dz \cap [y,z])$ has a source for each $D \in M$ for which it has an edge). It will suffice to show under these hypotheses that $\sharp(\Phi_{y,x}) = l(y,x)$ iff $\Omega(Dx \cap [y,x])$ has a source for each $D \in M$ for which it has an edge. Observe also that by induction, if $\Omega(Dx \cap [y,x])$ has an edge, it does not have a source iff $Dx \cap [y,x] = \{z_1, z_2, x\}$ where $z_i \xrightarrow{\alpha_i} x$ for $i = 1, 2$ (with $z_1 \neq z_2$).

Fix $z \in [y,x]$ with $z \xrightarrow{\gamma} x$. If $D \in M_\gamma$ with $l_D(z,x) > 1$, there is a directed path of length three or greater from z to x in $\Omega(Dx \cap [z,x])$ by Lemma 1.8. Let $D_1, D_2, \dots, D_m$ be all the distinct elements D of M_γ such that $\Omega(Dz \cap [y,z])$ has an edge or $l_D(z,x) > 1$. For each $i = 1, \dots, m$, $\Omega(D_i z \cap [y,x])$ has a source w_i by induction. For $p \in D_i z \cap [y,x]$ with $w_i \leq_{D_i} p$, we have $\{\, \alpha \in \Phi_{y,p} \mid s_\alpha \in D_i \,\} = \Phi^{D_i}_{w_i,p}$; this applies in particular with $p = x$ or z and so by our remarks on dihedral groups

$$\begin{aligned} \sharp(\{\alpha \in \Phi_{y,x} \mid s_\alpha \in D_i\}) &= l_{D_i}(y,x) = l_{D_i}(y,z) + l_{D_i}(z,x) \\ &= \sharp(\{\alpha \in \Phi_{y,z} \mid s_\alpha \in D_i\}) + l_{D_i}(z,x). \end{aligned} \tag{1.11.1}$$

Note $\gamma \in \Phi_{y,x}$ but $\gamma \notin \Phi_{y,z}$. Further,

$$1 + \sum_{D \in M_\gamma} (l_D(z,x) - 1) = l(z,x) \tag{1.11.2}$$

by [17, (1.2.1) and proof of 2.8]. By induction and definition of the D_i, this gives

$$\sum_i \sharp(\{\alpha \in \Phi_{y,z} \mid s_\alpha \in D_i\}) = \sharp(\Phi_{y,z}) = l(y,z).$$

Subtracting 1 from both sides of (1.11.1), summing over i and then adding 1 again therefore gives

$$\sharp(\{\alpha \in \Phi_{y,x} \mid s_\alpha \in \cup_i D_i\}) = l(y,x). \tag{1.11.3}$$

Now if $\sharp(\Phi_{y,x}) = l(y,x)$, then $\{\, s_\alpha \mid \alpha \in \Phi_{y,x} \,\} \subseteq \cup_i D_i$. This implies that if $Dx \cap [y,x] \supseteq \{z_1, z_2, x\}$ where $D \in M$, $z_i \xrightarrow{\alpha_i} x$ for $i = 1, 2$ and $z = z_1 \neq z_2$, then $D = D_i$ for some i and hence $Dx \cap [y,x] \neq \{z_1, z_2, x\}$ since $\Omega(D_i x \cap [y,x])$ has a source. On the other hand, suppose that for some $D \in M$, $Dx \cap [y,x] = \{z_1, z_2, x\}$ where $z_i \xrightarrow{\alpha_i} x$ for $i = 1, 2$ and $z_1 \neq z_2$. Taking $z = z_1$, $\gamma = \alpha_1$ above, we have $D \neq D_i$ for any i, $\alpha_2 \in \Phi_{y,x}$ but $s_{\alpha_2} \notin \cup_i D_i$, which implies by (1.11.3) that $\sharp(\Phi_{y,x}) \geq l(y,x) + 1$. □

1.12. We record here the conclusion of the main step of the above proof.

Corollary. *Assume $[y,x] \in \mathcal{P}$ is $\mathbb{R}$-smooth with $l(y,x) > 1$. Fix $y \leq z \xrightarrow{\gamma} x$. Let $D_1, D_2, \dots, D_m$ be all the distinct elements D of M_γ such that either $\Omega(Dz \cap [y,z])$ has an edge or $l_D(z,x) > 1$. Then $\Omega(D_i x \cap [y,x])$ has a source w_i, $\Phi_{y,x} = \cup_i \Phi^{D_i}_{w_i,x}$ and $\Phi_{y,z} = \cup_i \Phi^{D_i}_{w_i,z}$.*

1.13. Now we can prove a recursive formula for $c_{y,x}$ in case $[y,x]$ is $\mathbb{R}$-smooth.

Lemma. *Suppose $[y,x] \in \mathcal{P}$ is $\mathbb{R}$-smooth, with $l(y,x) \geq 1$. Fix $z \in [y,x]$ with $z \xrightarrow{\gamma} x$, and let $D_1, D_2, \dots, D_m$ (with $m \geq 0$) be be all the distinct elements D of M_γ such that either $\Omega(Dz \cap [y,z])$ has an edge or $l_D(z,x) > 1$. Then for each $i = 1, \dots, m$, $\Omega(D_i x \cap [y,x])$ has a unique source w_i and $c_{y,x} = c_{y,z} \prod_i \frac{c^{D_i}_{w_i,x}}{c^{D_i}_{w_i,z}}$ in $\mathbb{R}$.*

Proof. Take $B = S(V)$ in Lemma 1.9, so $B/\gamma B$ is a polynomial ring over $\mathbb{R}$ (and in particular it is a unique factorization domain). Then

$$S_{z,x} \equiv \frac{(-1)^{l(z)-l(x)}\prod_i c^{D_i}_{z,x}}{\gamma\prod_i\prod_{\alpha\in\Phi^{D_i}_{z,x}\setminus\{\gamma\}}\alpha}, \qquad S_{y,x} = \frac{(-1)^{l(y)-l(x)}c_{y,x}}{\prod_{\alpha\in\Phi_{y,x}}\alpha},$$
$$S_{y,z} = \frac{(-1)^{l(z)-l(y)}c_{y,z}}{\prod_{\alpha\in\Phi_{y,z}}\alpha}, \qquad S_{y,x} \equiv S_{y,z}S_{z,x}$$

where congruences are taken modulo $B_{(\gamma)}$ and we have used (1.11.2) for the sign in the formula for $S_{z,x}$. Substituting the first three of these equations in the fourth and clearing denominators gives

$$c_{y,x}\Big(\prod_{\alpha\in\Phi_{y,z}}\alpha\Big)\Big(\prod_{\alpha\in\cup_i\Phi^{D_i}_{z,x}\setminus\{\gamma\}}\alpha\Big) \equiv c_{y,z}\prod_i c^{D_i}_{z,x}\Big(\prod_{\alpha\in\Phi_{y,x}\setminus\{\gamma\}}\alpha\Big)$$

modulo $\gamma B_{(\gamma)}$. Since both sides above are in B, this holds even modulo γB (henceforward we write $\equiv$ for congruence modulo γB). The analogous result in D_i gives

$$c^{D_i}_{w_i,x}\Big(\prod_{\alpha\in\Phi^{D_i}_{w_i,z}}\alpha\Big)\Big(\prod_{\alpha\in\Phi^{D_i}_{z,x}\setminus\{\gamma\}}\alpha\Big) \equiv c^{D_i}_{w_i,z}c^{D_i}_{z,x}\Big(\prod_{\alpha\in\Phi^{D_i}_{w_i,x}\setminus\{\gamma\}}\alpha\Big)$$

for any $w_i \in D_i x$ such that $w_i \leq_{D_i} z$. Taking the product of the last equations over all i gives

$$\prod_i c^{D_i}_{w_i,x}\Big(\prod_{\alpha\in\cup_i\Phi^{D_i}_{w_i,z}}\alpha\Big)\Big(\prod_{\alpha\in\cup_i\Phi^{D_i}_{z,x}\setminus\{\gamma\}}\alpha\Big) \equiv \prod_i\Big(c^{D_i}_{w_i,z}c^{D_i}_{z,x}\Big)\Big(\prod_{\alpha\in\cup_i\Phi^{D_i}_{w_i,x}\setminus\{\gamma\}}\alpha\Big).$$

Comparing the last and third last equation above shows that

$$c_{y,z}\prod_i c^{D_i}_{w_i,x}\Big(\prod_{\alpha\in\Phi_{y,x}\setminus\cup_i\Phi^{D_i}_{w_i,x}}\alpha\Big) \equiv c_{y,x}\prod_i c^{D_i}_{w_i,z}\Big(\prod_{\alpha\in\Phi_{y,z}\setminus\cup_i\Phi^{D_i}_{w_i,z}}\alpha\Big). \tag{1.13.1}$$

If $[y,x]$ is $\mathbb{R}$-smooth, then taking w_i as the source of $\Omega(D_i x\cap[y,x])$, the proof is finished by Corollary 1.12 (congruence implies equality since both sides are in $\mathbb{R}$). □

1.14. Recall that A denotes a subring of $\mathbb{R}$ containing $\langle\alpha,\beta^\vee\rangle$ for all α, $\beta\in\Pi$.

Theorem. *Suppose that $[y,x]\in\mathcal{P}$ is $\mathbb{R}$-smooth. Then the following two conditions are equivalent:*

(i) *For each $z\in[y,x]$, $c_{y,z}$ is a unit in A.*
(ii) *For each $D\in M$ and $z\in[y,x]$ such that $\Omega(Dz\cap[y,z])$ has at least one edge, $c^D_{w,z}$ is a unit in A where w denotes the unique source of $\Omega(Dz\cap[y,z])$.*

If these conditions hold, we shall say that $[y,x]$ is A-smooth (this property depends on both the interval $[y,x]$ and the chosen reflection representation of W).

Proof. Write $A^\bullet$ for the group of units of A. We prove the equivalence by induction on $n = l(y,x)$. For $n\leq 1$, it is trivial. Suppose that $n\geq 2$ and the equivalence holds for intervals of length less than n. Consider an interval $[y,x]$ of length n such that $c_{y,z}\in A^\bullet$ for all $y\leq z<x$; equivalently by induction, for each such z and each $D\in M$ for which $\Omega(Dz\cap[y,z])$ has a source w_D, $c^D_{w_D,z}\in A^\bullet$. It will suffice to show that $c_{y,x}\in A^\bullet$ iff for each $D\in M$ for which $\Omega(Dx\cap[y,x])$ has a source w_D, $c^D_{w_D,x}\in A^\bullet$. Observe that for $D\in M$ such that $\Omega(Dx\cap[y,x])$ has a source

w_D, we necessarily have $l_D(w_D, x) \leq 1$ and hence $c^D_{w_D,x} = 1$, unless perhaps there is $y \leq z \xrightarrow{\gamma} x$ such that $D = D_j$, $w_D = w_j$ for some j (as in Lemma 1.13). But then for fixed such z we have $c_{y,x} \prod_i c^{D_i}_{w_i,z} = c_{y,z} \prod_i c^{D_i}_{w_i,x}$ where by induction, $c_{y,z} \in A^\bullet$, and each $c^{D_i}_{w_i,z} \in A^\bullet$. Thus, $c_{y,x} \in A^\bullet$ iff $c^{D_i}_{w_i,x} \in A^\bullet$ for all i. Since the choice of z with $y \leq z \xrightarrow{\gamma} x$ is arbitrary, this gives the desired equivalence. □

1.15. **Some Questions.** There is a close analogy between face lattices of convex polytopes and Chevalley-Bruhat intervals (cf [20], [11, Question 11.5]). We wish to state some questions and a conjecture suggested by this analogy and the results of this paper.

1.16. Fix an interval $[y, x] \in \mathcal{P}$ and $l(y, x) > 1$. For $z \in [y, x]$ with $z \xrightarrow{\gamma} x$, set $\Phi^z_{y,x} := \{\alpha \in \Phi_{y,x} | s_\alpha \in \cup^m_{i=1} D_i\}$ where $D_1, \ldots, D_m$ denote all the distinct elements D of M_γ such that either $\Omega(Dz \cap [y, z])$ has an edge or $l_D(z, x) > 1$. Consider the following condition $(1.16)_z$ on z:

$(1.16)_z$ $\quad$ $[y, z]$ is $\mathbb{R}$-smooth and $\Omega(D_i z \cap [y, z])$ has a source for $i = 1, \ldots, m$.

By the same argument as in the proof of (1.11.3), if $(1.16)_z$ holds then $\sharp(\Phi^z_{y,x}) = l(y, x)$. By Lemma 1.11, $[y, x]$ is $\mathbb{R}$-smooth iff $(1.16)_z$ holds for each $y \leq z \xrightarrow{\gamma} x$; moreover, in that case $\Phi^z_{y,x} = \Phi_{y,x}$ for each such z. It follows that $[y, x]$ is $\mathbb{R}$-smooth iff for all $z \in [y, x]$ with $z \xrightarrow{\gamma} x$, $[y, z]$ is $\mathbb{R}$-smooth and $\Phi^z_{y,x} = \Phi_{y,x}$ (cf [6, Theorem 1.1,1.2]).

Question. Suppose $l(y, x) \geq 2$. Is $[y, x]$ $\mathbb{R}$-smooth iff there are z_i in $[y, x]$ satisfying $z_i \xrightarrow{\alpha_i} x$ and the condition $(1.16)_{z_i}$, with $\Phi^{z_1}_{y,x} = \Phi^{z_2}_{y,x}$ and $z_1 \neq z_2$?

1.17. It can be shown that in special cases, $S_{y,x}$ is a degeneration of a rational function whose values describe volumes of frustums of a polyhedral cone associated to $[y, x]$. We do not know if it is reasonable to expect this to be true in general, but at least conjecture that the following consequence of such a description holds.

Conjecture. For any $y \leq x$, $S_{y,x}$ expressible in the form $S_{y,x} = \sum_\Psi c_\Psi \prod_{\alpha \in \Psi} \alpha^{-1}$ for certain $c_\Psi \in \mathbb{R}_{\geq 0}$, where the sum ranges over subsets Ψ of $\Phi_{y,x}$ of cardinality $l(y, x)$.

One could ask if this true even requiring $c_\Psi \in A \cap \mathbb{R}_{\geq 0}$.

1.18. One might wonder if there is some sense in which Chevalley-Bruhat intervals themselves may be described using "rank two data." As an example of a naive question in this direction, one could ask the following:

Question. Let $z, z' \in [y, x]$. Is it true that $z \leq z'$ iff for each $D \in M$, $Dx \cap [z', x] \subseteq Dx \cap [z, x]$ and $Dy \cap [y, z] \subseteq Dy \cap [y, z']$.

We mention that for W of type A_4 and $[y, x] = W$, it is not true that $z \leq z'$ iff for each $D \in M$, $Dx \cap [z', x] \subseteq Dx \cap [z, x]$.

1.19. **More general orders.** The previous results and questions apply mutatis mutandis to Chevalley Bruhat order and to the orders on W defined in [17]. This is mostly a matter of regarding the length function l on which the preceding definitions implicitly depend as a "parameter," as we now indicate.

Let T denote the set of reflections of W and $\mathcal{P}(T)$ denote the power set of T regarded as abelian group under symmetric difference. Let $N \colon W \to \mathcal{P}(T)$ be the "reflection cocycle" given by $N(w) = \{t \in T \mid l'(tw) < l'(w)\}$ where l' denotes the standard length function. We let $\mathcal{I}$ be the set of initial sections I of reflection orders of T; $\mathcal{I}$ has a W action $w \cdot I = N(w) + wIw^{-1}$ (see [17], [18] for details here and below). For example, the collection of subsets $\{s_\alpha \mid \alpha \in \Phi_+, \alpha \succeq 0\}$ as $\succeq$ ranges over the vector space total orders of V forms a W-invariant subset of $\mathcal{I}$.

Fix $I \in \mathcal{I}$. There is then a corresponding length function $l^I = l^I_W : W \to \mathbb{Z}$ defined by $l^I(w) = l'(w) - 2\sharp(N(w^{-1}) \cap I))$ where l' is the standard length function. Set $l^I(v,w) = l^I(w) - l^I(v)$. Write $v \xrightarrow{\alpha}{}^{I}_{W} w$ if $\alpha \in \Phi_+$ and $w = s_\alpha v$ with $l^I(v) < l^I(w)$. Define $\Omega^I = \Omega^I_W$ as the edge-labeled directed graph with edges $v \xrightarrow{\alpha}{}^{I}_{W} w$, and $\leq^I_W = \leq^I$ as the partial order on W with $v \leq^I w$ iff there is a directed path from v to w in Ω^I. Finally, set $\Phi^I_{z,w} = \Phi^{W,I}_{z,w} = \{\, \alpha \in \Phi_+ \mid z \leq^I v \xrightarrow{\alpha}{}^{I}_{W} w \,\}$ and write $[z,w]^I = [z,w]^I_W = \{\, v \in W \mid z \leq^I v \leq^I w \,\}$. Note that the undirected labeled graphs underlying Ω_W and Ω^I_W coincide. We write $\mathcal{P}^I$ for the set of non-empty finite intervals $[y,x]^I_W$ in which every length two closed subinterval has cardinality four (for many I of interest, $\mathcal{P}^I$ is the set of all closed intervals of W in $\leq^I$).

1.20. From [12], there are elements $S^I_{w,z}$ of $Q(V)$, defined for $w, z \in W$ with either $[w,z] \in \mathcal{P}^I$ or $w \not\leq^I z$, satisfying the recurrence formulae Lemma 1.5(i)–(iv). (These are constructed from elements $\hat{S}^I_{z,w}$ which have a natural description similar to that for the $\hat{S}_{z,w}$, involving a module constructed from $\leq^I$ for the nil Hecke ring). The following relation for $[z,w] \in \mathcal{P}^I$ is established in loc cit:

$$(1.20.1) \qquad \sum_{v \in [z,w]^I} (-1)^{l(v)-l(w)} S^I_{z,v} S^{T+I}_{w,v} = \delta_{z,w} = \sum_{v \in [z,w]^I} (-1)^{l(v)-l(z)} S^I_{v,w} S^{T+I}_{v,z}$$

i.e. the "matrices" $(S^I_{z,w})_{z,w}$ and $((-1)^{l(z)-l(w)} S^{T+I}_{w,z})_{z,w}$ are mutually inverse. Since $l^I(v)$ and $l(v)$ have the same parity, l could be replaced by l^I if desired in these equations. We have in particular $S_{z,w} = S^T_{z,w}$ above.

1.21. Define $c^I_{z,w} = S^I_{z,w} \prod_{\alpha \in \Phi^I_{z,w}} \alpha$ for any $[z,w] \in \mathcal{P}^I$. For any reflection subgroup D of W and any $[z,w] \in \mathcal{P}^I$ with $Dz = Dw$, define $S^{D,I}_{z,w}$, $\Phi^{D,I}_{z,w}$, $c^{D,I}_{w,z}$ as follows; let p be any element of Dz, set $J = p \cdot I \cap D$ and define $S^{D,I}_{z,w} := S^{D,J}_{zp^{-1},wp^{-1}}$, $\Phi^{D,I}_{z,w} := \Phi^J_{zp^{-1},wp^{-1}}$, $c^{D,I}_{z,w} := c^J_{zp^{-1},wp^{-1}}$ where the right hand sides are computed in D (with its natural based root datum and order $\leq^J$). One needs to note that $[zp^{-1}, wp^{-1}]^J_D \in \mathcal{P}^J_D$ if $[z,w]^I \in \mathcal{P}^I_W$, from [17, (2.2)]. These definitions are independent of the choice of p, using the analogue [17, 1.4(3)] of Lemma 1.8.

Theorem. Theorem 1.6, Lemma 1.9, Theorem 1.11, Corollary 1.12, Lemma 1.13 *and* Theorem 1.14 *hold mutatis mutandis for the corresponding objects associated to l^I i.e. if we replace l by l^I, Ω by Ω^I, $\leq$ by $\leq^I$, $\to$ by $\to^I_W$, $\Phi_{z,w}$ by $\Phi^I_{z,w}$, $\mathcal{P}$ by $\mathcal{P}^I$, $[z,w]$ by $[z,w]^I_W$, $S_{z,w}$ by $S^I_{z,w}$, $c_{z,w}$ by $c^I_{z,w}$, and $c^D_{z,w}$ by $c^{D,I}_{z,w}$.*

Proof. The generalization of 1.6 to orders $\leq^I$ is proved in [12]. The Z-property holds for $\leq^I$, but for the inductive proofs one requires as well a recursive characterization [17, (2.5)] of $\mathcal{P}^I$. We supplement the remarks on dihedral groups in 1.10 by the following observations. For a dihedral Coxeter group W' (with 2 simple reflections and reflections T') and any initial section J of a reflection order on T', $\Omega^J_{W'}$ is either isomorphic to $\Omega^{T'}_{W'}$ where T' is the set of reflections of W', or to $\Omega^{\emptyset}_{W'}$ or, for infinite W', one other possibility for which $(W', \leq^J_{W'})$ is order isomorphic to $\mathbb{Z}$ in its usual total order. In the latter case, the only intervals $[z,w]^J_{W'} \in \mathcal{P}^J_{W'}$ have either $z=w$ or $[z,w]^J_{W'}=\{z,w\}$ with $z \xrightarrow{\alpha}{}^J_{W'} w$ for some $\alpha \in \Phi_+$. Because of [17, (2.2) and (2.5)], it is still true that for $[y,x]^I_W \in \mathcal{P}^I_W$, $z \in [y,x]^I_W$ and $D \in M$, the unlabeled graph underlying $\Omega([y,x]^I_W \cap Dz)$ is isomorphic to $G(X)$ for some interval X in the order $\preceq$ on the graph G considered in 1.10. Taking the above remarks into account, the proofs extend mutatis mutandis from the case of l to l^I. □

1.22. We find it convenient to use the following terminology. The A-singular locus of $y \in W$ in the order $\leq^I$ is defined to be the set of all $x \in W$ such that $[y,x] \in \mathcal{P}^I$ and $[y,x]^I$ is A-singular (i.e. not A-smooth in the sense of Theorem 1.14); this notion depends on y, I and the root system. If the A-singular locus is empty, we shall say y is A-smooth for $\leq^I$. We shall also say that x is A-smooth (resp., A-singular) in y in the order $\leq^I$ if $[y,x]^I$ is (resp., is not) A-smooth.

2. Rank Two Results

In this section, we explicitly compute the rational functions $S_{x,w}$ for dihedral groups W and discuss divisibility properties of their numerators, which turn out to be binomial coefficients modeled on sequences of root coefficients. The sequence of root coefficients has of course been previously studied; see for example [25], [15] and [14, Ch 2]. These root coefficients are also intimately related to many classical objects including Gaussian integers, Chebyshev functions, cyclotomic polynomials etc and variants of many of the formulae given here have appeared in the literature in these (and other) contexts; we indicate some of these connections but make no attempt to supply full references or discuss the relationships comprehensively.

2.1. Throughout this section, we use the following notational convention. For elements r,s of a monoid M, we write

$$(rs)_n := \begin{cases} r_1 r_2 \dots r_n, & \text{if } n>0 \\ 1_M, & \text{if } n=0 \end{cases}$$

where $r_{2i+1}=r$ and $r_{2i}=s$. If r, s are invertible, we extend this to define $\{(rs)_n\}_{n\in\mathbb{Z}}$ so $(rs)_n=(s^{-1}r^{-1})_{-n}$ for all $n \in \mathbb{Z}$. If r, s are involutions (i.e. $r^2=s^2=1$), then

$$(rs)_n(rs)_m=(rs)_n(sr)_{-m}=(rs)_{n+(-1)^n m}.$$

2.2. **Root coefficients.** For a commutative ring R with identity element 1_R and elements $a,b \in R$, define sequences $\{c_n\}_{n\in\mathbb{Z}}$, $\{d_n\}_{n\in\mathbb{Z}}$ in R by

$$c_0=0, \quad c_1=1_R, \quad c_{n-1}+c_{n+1}=a_n c_n \text{ for } n \in \mathbb{Z} \tag{2.2.1}$$

$$d_0=0, \quad d_1=1_R, \quad d_{n-1}+d_{n+1}=b_n d_n \text{ for } n \in \mathbb{Z} \tag{2.2.2}$$

where $a_{2m+1}=b_{2m}=a$ and $a_{2m}=b_{2m+1}=b$ for $m \in \mathbb{Z}$. If necessary, we write $c_n=c_n(a,b)$ and $d_n=d_n(a,b)$ to indicate dependence of c_n and d_n on a and b, so for a homomorphism $\phi\colon R \to R'$ of commutative rings, $\phi(c_n(a,b))=c_n(\phi(a),\phi(b))$

and $\phi(d_n(a,b)) = d_n(\phi(a),\phi(b))$. We have

$$d_n(a,b) = c_n(b,a), \qquad c_n(-a,-b) = (-1)^{n+1}c_n(a,b) \tag{2.2.3}$$

$$c_{-n} = -c_n, \qquad \begin{cases} c_n = d_n, & \text{if } n \text{ is odd} \\ bc_n = ad_n, & \text{if } n \text{ is even} \end{cases} \tag{2.2.4}$$

$$c_n(2,2) = n \quad \text{in } R \tag{2.2.5}$$

$$(c_n)_{n=0}^6 = (0, 1, a, ab-1, a^2b-2a, a^2b^2-3ab+1, a^3b^2-4a^2b+3a). \tag{2.2.6}$$

2.3. Let R be a commutative ring. Let F be a free group of rank two with generators r, s. Let V (resp., V') be a R-module containing elements $\{A,B\}$ (resp., $\{A^\vee,B^\vee\}$). Suppose given a fixed (possibly degenerate) R-bilinear map $\langle\ ,\ \rangle\colon V\times V'\to R$ satisfying

$$\langle A,A^\vee\rangle = \langle B,B^\vee\rangle = v+v^{-1}, \qquad \langle A,B^\vee\rangle = -b, \qquad \langle B,A^\vee\rangle = -a$$

where a, b, v are elements of R with v invertible in R (in this paper, we use only the case $v=1$). If $\{A,B\}$ is a basis of V and $\{A^\vee,B^\vee\}$ is a basis of V', there is a unique such bilinear form, but we do not assume these conditions.

Define an action of F on V (resp., V') by setting

$$r(z) = vz - \langle z,A^\vee\rangle A, \qquad s(z) = vz - \langle z,B^\vee\rangle B \qquad \text{for } z\in V \tag{2.3.1}$$

$$r(z) = v^{-1}z - \langle A,z\rangle A^\vee, \qquad s(z) = v^{-1}z - \langle B,z\rangle B^\vee \qquad \text{for } z\in V'. \tag{2.3.2}$$

We have for instance $r^{-1} = r-(v-v^{-1})\mathrm{Id}_V$ on V and $r^{-1} = r+(v-v^{-1})\mathrm{Id}_{V'}$ on V' (r, s afford representations of appropriate, possibly different, Iwahori-Hecke algebras of an infinite dihedral group on V and V'). Note

$$\langle fz,z'\rangle = \langle z,f^{-1}z'\rangle \qquad \text{for } z\in V,\ z'\in V' \text{ and } f\in F. \tag{2.3.3}$$

2.4. If $\{A,B\}$ is a basis of V, identify $\mathrm{GL}(V)$ with $\mathrm{GL}_2(R)$ by taking matrices of R-endomorphisms of V, with respect to the ordered basis A,B of V. Then $r = \left(\begin{smallmatrix} -v^{-1} & a \\ 0 & v\end{smallmatrix}\right)$, $s = \left(\begin{smallmatrix} v & 0 \\ b & -v^{-1}\end{smallmatrix}\right)$. By induction and the definition of $c_n = c_n(a,b)$ and $d_n = d_n(a,b)$,

$$(rs)_{2n} = \begin{pmatrix} c_{2n+1} & -v^{-1}c_{2n} \\ vd_{2n} & -d_{2n-1}\end{pmatrix}, \qquad (rs)_{2n+1} = \begin{pmatrix} -v^{-1}c_{2n+1} & c_{2n+2} \\ -d_{2n} & vd_{2n+1}\end{pmatrix}, \tag{2.4.1}$$

$$(sr)_{2n} = \begin{pmatrix} -c_{2n-1} & vc_{2n} \\ -v^{-1}d_{2n} & d_{2n+1}\end{pmatrix}, \qquad (sr)_{2n+1} = \begin{pmatrix} vc_{2n+1} & -c_{2n} \\ d_{2n+2} & -v^{-1}d_{2n+1}\end{pmatrix}. \tag{2.4.2}$$

Returning to the general situation in which it is not assumed that V has $\{A,B\}$ as basis, there are similar formulae $\left(\begin{smallmatrix} r(A) \\ r(B)\end{smallmatrix}\right) = \left(\begin{smallmatrix} -v^{-1} & 0 \\ a & v\end{smallmatrix}\right)\left(\begin{smallmatrix} A \\ B\end{smallmatrix}\right)$ etc.

2.5. Define "roots" $\{\alpha_n\}_{n\in\mathbb{Z}}$ and $\{\beta_n\}_{n\in\mathbb{Z}}$ in V by

$$\alpha_n := (rs)_{n-1}(A_n), \qquad \beta_n := (sr)_{n-1}(B_n). \tag{2.5.1}$$

where $A_{2m} = B_{2m+1} = B$ and $A_{2m+1} = B_{2m} = A$. From the above matrices,

$$\alpha_n = c_nA + vd_{n-1}B, \qquad \beta_n = d_nB + vc_{n-1}A \tag{2.5.2}$$

and so α_n, β_n are determined by recursive formulae

$$\alpha_0 = -vB, \quad \alpha_1 = A, \quad \alpha_{n-1}+\alpha_{n+1} = a_n\alpha_n \tag{2.5.3}$$

$$\beta_0 = -vA, \quad \beta_1 = B, \quad \beta_{n-1}+\beta_{n+1} = b_n\beta_n. \tag{2.5.4}$$

Similarly, define "coroots" $\{\alpha_n^\vee\}_{n\in\mathbb{Z}}$ and $\{\beta_n^\vee\}_{n\in\mathbb{Z}}$ in V' by

$$\alpha_n^\vee := (rs)_{n-1}(A_n^\vee) = v^{-1}c_{n-1}B^\vee + d_nA^\vee, \tag{2.5.5}$$

$$\beta_n^\vee := (sr)_{n-1}(B_n^\vee) = v^{-1}d_{n-1}A^\vee + c_nB^\vee. \tag{2.5.6}$$

2.6. In this subsection, we assume that $v = 1$. Then the actions of F on V and V' factor through the infinite dihedral group $W := F/\langle r^2, s^2\rangle$, and $\beta_n = -\alpha_{-n+1}$ for any $n \in \mathbb{Z}$. For integers p and n

$$(rs)_n\alpha_p = \begin{cases}\alpha_{p+n} & \text{if } n \text{ is even}\\ \beta_{p-n} & \text{if } n \text{ is odd}\end{cases}, \qquad (sr)_n\alpha_p = \begin{cases}\alpha_{p-n} & \text{if } n \text{ is even}\\ \beta_{p+n} & \text{if } n \text{ is odd}\end{cases} \tag{2.6.1}$$

(by symmetry, these equations reduce immediately to their easily checked special case in which $n = 1$). Using this, one checks that for integers m and n,

$$\langle\, \alpha_m, \alpha_n^\vee \,\rangle = \begin{cases} c_{n-m+1} - c_{n-m-1} & m \text{ even or } n \text{ odd}\\ d_{n-m+1} - d_{n-m-1} & m \text{ odd or } n \text{ even.}\end{cases} \tag{2.6.2}$$

In particular, $\langle \alpha_p, \alpha_p^\vee\rangle = 2$. Moreover, using (2.3.3), the reflections $(rs)_{2p-1}$ of W act on V and V' according to

$$(rs)_{2p-1}(z) = z - \langle z, \alpha_p^\vee\rangle\alpha_p \quad \text{for } z \in V \tag{2.6.3}$$

$$(rs)_{2p-1}(z) = z - \langle \alpha_p, z\rangle\alpha_p^\vee \quad \text{for } z \in V'. \tag{2.6.4}$$

Lemma. *Fix integers m, n and set $t := m + n$. Define $a' := \langle\, \alpha_{-m}, \alpha_n^\vee \,\rangle$ and $b' := \langle\, \alpha_n, \alpha_{-m}^\vee \,\rangle$ i.e.*

$$(a', b') = \begin{cases}(c_{t+1} - c_{t-1}, d_{t+1} - d_{t-1}) & \textit{if } m \textit{ is even or } n \textit{ is odd}\\ (d_{t+1} - d_{t-1}, c_{t+1} - c_{t-1}) & \textit{if } m \textit{ is odd or } n \textit{ is even.}\end{cases}$$

Abbreviate $c'_n := c_n(a', b')$ and $d'_n = d_n(a', b')$. Then

(a) *If t is even, $c_{t+1} - c_{t-1} = d_{t+1} - d_{t-1}$, and if t is odd, $b(c_{t+1} - c_{t-1}) = a(d_{t+1} - d_{t-1})$.*
(b) $\det\left(\begin{smallmatrix} 2 & -a' \\ -b' & 2\end{smallmatrix}\right) = c_td_t \det\left(\begin{smallmatrix} 2 & -a \\ -b & 2\end{smallmatrix}\right)$
(c) *For any $l \in \mathbb{Z}$, $c_{ln+(l-1)m} = c'_lc_n + d'_{l-1}c_m$ and $d_{ln+(l-1)m} = d'_ld_n + c'_{l-1}d_m$.*

Proof. Part (a) follows immediately using (2.2.4). Define $\tilde{A} = \alpha_n$, $\tilde{A}^\vee = \alpha_n^\vee$, $\tilde{B} = -\alpha_{-m}$, and $\tilde{B}^\vee = -\alpha_{-m}^\vee$. The matrix on the right (resp., left) side of (b) is $(\langle Y, X^\vee\rangle)_{X,Y\in\{A,B\}}$ (resp., $(\langle Y, X^\vee\rangle)_{X,Y\in\{\tilde{A},\tilde{B}\}}$). We may define $\tilde{r}$, $\tilde{s}$, $\tilde{\alpha}_n$, $\tilde{\beta}_n$, $\tilde{\alpha}_n^\vee$ etc exactly as r, s, α_n, $\alpha_n^\vee$, again using $\langle\ ,\ \rangle\colon V \times V' \to R$ (with $v = 1$) but using $\tilde{A}$, $\tilde{A}^\vee$, $\tilde{B}$ and $\tilde{B}^\vee$ in place of A, $A^\vee$, B and $B^\vee$ respectively. Therefore, $\tilde{\alpha}_s = c'_sA' + d'_{s-1}B'$. On the other hand, from above we have $\tilde{r} = (rs)_{2n-1}$, $\tilde{s} = (rs)_{-2m-1}$, $(\tilde{r}\tilde{s})_{2k} = (rs)_{2kt}$ and so

$$\tilde{\alpha}_{2k+1} = (\tilde{r}\tilde{s})_{2k}(\tilde{\alpha}_1) = (rs)_{2kt}(\alpha_n) = \alpha_{2km+(2k+1)n}$$
$$\tilde{\alpha}_{2k} = (\tilde{r}\tilde{s})_{2k}(\tilde{\alpha}_0) = (rs)_{2kt}(\alpha_{-m}) = \alpha_{(2k-1)m+2kn}.$$

The first identity in (c) follows by taking A, B linearly independent in V, as we may, and equating coefficients of A in

$$\begin{aligned} c_{ln+(l-1)m}A + d_{ln+(l-1)m-1}B &= \alpha_{ln+(l-1)m} = \tilde{\alpha}_l \\ &= c'_l\tilde{A} + d'_{l-1}\tilde{B} = c'_l(c_nA + d_{n-1}B) - d'_{l-1}(c_{-m}A + d_{-m-1}B). \end{aligned}$$

Equating coefficients of B and then replacing m by $m+1$ and n by $n-1$ gives the identity $d_{ln+(l-1)m} = c_l(b',a')d_n + d_{l-1}(b',a')d_m$ i.e. the second identity in (c).

For the proof of (b), we need a special case of the identities in (c). For integers p and q define

$$F_{p,q} := c_p d_{q-1} - d_{p-1}c_q, \qquad G_{p,q} := d_p c_{q-1} - c_{p-1}d_q$$

in R. Now if in (c) one takes $n = 1-m$, $t = 1$ (so $(a',b') = (a,b)$ for even m and $(a',b') = (b,a)$ for odd m), one gets identities equivalent to the following:

$$F_{p,q} = c_p d_{q-1} - d_{p-1}c_q = \begin{cases} d_{q-p}, & \text{if } p \text{ or } q \text{ is odd} \\ c_{q-p}, & \text{if } p \text{ or } q \text{ is even.} \end{cases} \tag{2.6.5}$$

By symmetry between F and G, this implies that

$$F_{p+2k,q+2k} = F_{p,q}, \qquad G_{p+2k,q+2k} = G_{p,q}, \qquad G_{p,q} = F_{p+1,q+1}. \tag{2.6.6}$$

To prove (b), we show $abc_td_t - a'b' - 4c_td_t = -4$ as follows:

$$\begin{aligned}(c_{t+1}+c_{t-1})(d_{t+1}+d_{t-1}) - (c_{t+1}-c_{t-1})(d_{t+1}-d_{t-1}) - 4c_td_t = \\ 2(c_{t-1}d_{t+1} + c_{t+1}d_{t-1} - 2c_td_t) = 2(F_{t+1,t} + G_{t+1,t}) = 2(c_{-1}+d_{-1}) = -4.\end{aligned}$$

□

2.7. Let $F_{l,m}$ be as defined in the proof of Lemma 2.6.

Lemma. *For $l,m,n \in \mathbb{Z}$,*

(a) $F_{l,m}\alpha_n + F_{m,n}\alpha_l + F_{n,l}\alpha_m = 0$.
(b) $F_{l+1,m+1}\beta_n + F_{m+1,n+1}\beta_l + F_{n+1,l+1}\beta_m = 0$.

Proof. Observe first that

$$F_{q,p} = -F_{p,q}, \qquad F_{p+1,q} + F_{p-1,q} = a_pF_{p,q}. \tag{2.7.1}$$

The equation in (a) may be rewritten as $F_{l,n}\alpha_m = F_{m,n}\alpha_l - F_{m,l}\alpha_n$. Now for fixed l and n, this holds for both $m = M$ and $m = M-1$ iff it holds for both $m = M$ and $m = M+1$, by (2.7.1) and (2.5.3). By the invariance of the equation (a) under cyclic permutations of (l,m,n), it is therefore enough to prove (a) for $l,m,n \in \{0,1\}$. But then, say $m = n$ and $F_{l,m}\alpha_n + F_{m,n}\alpha_l + F_{n,l}\alpha_m = F_{l,n}\alpha_n + 0\alpha_l - F_{l,n}\alpha_n = 0$. The equation in (b) follows by symmetry and (2.6.6). □

2.8. Here, we record some further consequences of Lemma 2.6(c). Replacing m by $-m$, the first identity there may be rewritten

$$c_{m+lt} = c_l(a',b')c_{m+t} - d_{l-1}(a',b')c_m, \tag{2.8.1}$$

$$(a',b') = \begin{cases} (c_{t+1}-c_{t-1}, d_{t+1}-d_{t-1}) & \text{if } m \text{ is even} \\ (d_{t+1}-d_{t-1}, c_{t+1}-c_{t-1}) & \text{if } m \text{ is odd} \end{cases} \tag{2.8.2}$$

for integers m, t, l. Replacing l by $-l$ and adding gives

$$c_{m+lt} + c_{m-lt} = \big(d_{l+1}(a',b') - d_{l-1}(a',b')\big)c_m \tag{2.8.3}$$

A telescoping sum argument shows that for $N \in \mathbb{Z}$ and $t,q \in \mathbb{N}$.

$$c_N + c_{N+2t} + \cdots + c_{N+2t(q-1)} + c_{N+2tq} = d_{q+1}(a',b')c_{N+tq} \tag{2.8.4}$$

where (a', b') are as above with $m = n + tq$. Taking $l = 1$ in (2.8.3) gives

$$c_{m+t} + c_{m-t} = \begin{cases} (c_{t+1} - c_{t-1})c_m & \text{if } m \text{ is even} \\ (d_{t+1} - d_{t-1})c_m & \text{if } m \text{ is odd.} \end{cases} \tag{2.8.5}$$

From (2.6.5), we have

$$c_{n+m} = \begin{cases} d_{m+1}c_n - d_{n-1}c_m & \text{if } n \text{ or } m \text{ is even} \\ c_n c_{m+1} - c_{n-1}c_m & \text{if } n \text{ and } m \text{ are odd} \end{cases} \tag{2.8.6}$$

where for the case of odd n and m, we use $c_n = d_n$, $c_m = d_m$. On the other hand, taking $m = 0$ in 2.6(c) gives

$$c_{ln} = c_l(a'', b'')c_n, \qquad d_{ln} = d_l(a'', b'')d_n \tag{2.8.7}$$

where $(a'', b'') = (c_{n+1} - c_{n-1}, d_{n+1} - d_{n-1})$.

2.9. It is convenient to collect here the following miscellaneous facts.

Lemma. (a) *For integers m, n, g with $\gcd(m, n) = g$, we have $Rc_m + Rc_n = Rc_g$.*

(b) *If $c_l = 0$, then $c_{l+k} = -c_{l-k}$ and $c_{k+2l} = c_k$.*

(c) *We have $\{\, m \mid c_m = 0 \,\} = \mathbb{Z}l$ and $\{\, m \mid d_m = 0 \,\} = \mathbb{Z}l'$ for some $l, l' \in \mathbb{N}$.*

(d) *If R is finite, we have $1 \le l, l' \le 2|R|^2$ where l, l' are as in* (c).

For (e)–(i), *assume R is an integral domain and let l, l' be as in* (c).

(e) *If $l \ne l'$, then one of l, l' is equal to 2 and the other is even.*

(f) *If $l > 0$ is even, then $c_{m+kl} = (-1)^k c_m$ for $m, k \in \mathbb{Z}$, and $c_{l-1} = d_{l-1} = 1$.*

(g) *If l is odd, then $d_{m+l} = d_{l+1}c_m$ and $c_{m+l} = c_{l+1}d_m$ for $m \in \mathbb{Z}$, with $c_{l+1}d_{l+1} = 1_R$.*

(h) *If l is odd and $a = b$, then for $m, k \in \mathbb{Z}$, $c_m = d_m$ and $c_{m+kl} = c_{l+1}^k c_m$ with $c_{l+1} \in \{\pm 1\}$.*

(i) *If $l > 0$, then $c_{\pm 1+kl}$ and $d_{\pm 1+kl}$ are units of R for all integers k.*

Proof. The identities (2.8.7) and (2.8.6) imply that whenever we have an identity $s = xm + yn$ in $\mathbb{Z}$, there is a corresponding identity $c_s = Xc_m + Yc_n$ where X, Y are certain elements of R. Part (a) is an immediate consequence of this and the fact that $\mathbb{Z}$ is a principal ideal domain, while (b) follows from (2.8.5) and (c) follows from (b) (or (a)). For (d), note there are integers $0 \le N < M \le |R|^2$ with $(c_{2N}, c_{2N+1}) = (c_{2M}, c_{2M+1}) \in R \times R$. The definitions give $c_{2N+k} = c_{2M+k}$ for all $k \in \mathbb{Z}$; in particular, $c_{2(M-N)} = c_0 = 0$, proving (d). Now we assume R is an integral domain, and that l, l' are as in (c). If $l \ne 2$ (i.e. $a \ne 0$), then $c_n = 0$ implies $d_n = 0$, by (2.2.4). Similarly, if $l' \ne 2$, then $d_n = 0$ implies $c_n = 0$. Part (e) follows readily using (c). By (2.6.5), we have

$$c_m d_{l-1} = \begin{cases} d_{l-m} & \text{if } l \text{ is odd} \\ c_{l-m} & \text{if } l \text{ is even} \end{cases} \tag{2.9.1}$$

If $l > 0$ is even, we get $c_m d_{l-1} = -c_{l+m}$; taking $m = -l/2$ gives $(d_{l-1} - 1)c_{l/2} = 0$, hence $d_{l-1} = 1$ and (f) follows. If $l > 0$ is odd, we get $l' = l$ by (e) and $c_m d_{l+1} = d_{l+m}$, $d_m c_{l+1} = c_{l+m}$ so $c_{l+1}d_{l+1}c_m = c_{l+1}d_{l+m} = c_{2l+m} = c_m$ and (g) follows. Finally, (h) is immediate from (g) and (i) follows from (f)–(g). $\square$

Remarks. Of course, we get analogous results to those stated above by interchanging a and b, c and d, and l and l'. The above Lemma implies that if $c_f = d_f = 0$ for some $f > 0$ and A, B form a R-basis of V, then r, s satisfy the braid relation $(rs)_f = (sr)_f$ on V and so afford a representation (essentially the well known reflection representation) of an Iwahori-Hecke algebra of a finite dihedral group of order $2f$.

2.10. **Binomial Coefficients.** Till 2.24, we take $R = \mathbb{Z}[x, y]$, the integral polynomial ring in two variables x and y. We write $c_n = c_n(x, y)$, $d_n = d_n(x, y)$ as in 2.2. Fix an algebraically closed field F containing $\mathbb{Z}[x, y]$ as a subring. We consider in F the elements

$$(2.10.1) \qquad \gamma := \sqrt{x}\sqrt{y}, \qquad t := \frac{\gamma + \sqrt{\gamma^2 - 4}}{2}, \qquad v := \frac{\sqrt{x}}{\sqrt{y}}t, \qquad w := \frac{\sqrt{y}}{\sqrt{x}}t$$

where arbitrary fixed square roots of x, y, $\gamma^2 - 4$ have been chosen in F. We have

$$(2.10.2) \qquad \gamma = t + t^{-1}, \qquad v + w^{-1} = x, \qquad v^{-1} + w = y, \qquad vw = t^2.$$

In particular, v and w are algebraically independent over $\mathbb{Q}$, t is transcendental over $\mathbb{Q}$ and $\mathbb{Z}[x, y]$ is a subring of the ring $\mathbb{Z}[v^{\pm 1}, w^{\pm 1}]$ of Laurent polynomials in v and w over $\mathbb{Z}$. Define elements $\gamma_n \in F$ as follows (see (2.2.4)):

$$(2.10.3) \qquad \gamma_n := \begin{cases} \frac{\sqrt{y}}{\sqrt{x}} c_n = \frac{\sqrt{x}}{\sqrt{y}} d_n & \text{if } n \text{ is even} \\ c_n = d_n & \text{if } n \text{ is odd.} \end{cases}$$

Then the equation (2.2.1) becomes

$$(2.10.4) \qquad \gamma_0 = 0, \qquad \gamma_1 = \gamma, \qquad \gamma_{n-1} + \gamma_{n+1} = \gamma\gamma_n.$$

It follows that γ_n is the n-th Gaussian integer in $\mathbb{Z}[t, t^{-1}]$ (cf [26]):

$$(2.10.5) \qquad \gamma_n = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}] \qquad \text{for } n \in \mathbb{Z}.$$

The above equations also imply that

$$(2.10.6) \qquad c_n = \begin{cases} v\frac{(vw)^{n/2} - (vw)^{-n/2}}{vw - 1} & \text{if } n \text{ is even} \\ \frac{(vw)^{(n+1)/2} - (vw)^{-(n-1)/2}}{vw - 1} & \text{if } n \text{ is odd,} \end{cases}$$

with d_n given similarly (interchanging v and w) i.e. c_n, d_n are "two-parameter Gaussian integers" as elements of $\mathbb{Z}[v^{\pm 1}, w^{\pm 1}]$.

2.11. For any integer $N \geq 1$, let $\phi_N \in \mathbb{Z}[t]$ denote the N-th cyclotomic polynomial. Let φ denote the Euler totient function, so $\varphi(N) = \deg \phi_N$. Recall that

$$(2.11.1) \qquad t^n - 1 = \prod_{\substack{N|n \\ N \geq 1}} \phi_N; \qquad \text{if } N \text{ is odd then } \phi_{2N}(t) = \phi_N(-t);$$

$$(2.11.2) \qquad \text{For } p \text{ prime, } \phi_N(t^p) = \begin{cases} \phi_{pN}(t) & \text{if } p|N \\ \phi_{pN}(t)\phi_N(t) & \text{otherwise.} \end{cases}$$

Note that $\mathbb{Z}[\gamma]$ is the subring of $\mathbb{Z}[t, t^{-1}]$ consisting of elements fixed by the ring involution θ of $\mathbb{Z}[t, t^{-1}]$ determined by $\theta(t^k) = t^{-k}$. Define monic polynomials Γ_N in $\mathbb{Z}[\gamma]$ by $\Gamma_N := t^{-\varphi(N)}\phi_N(t^2) = t^{\varphi(N)}\phi_N(t^{-2})$ for $N \geq 2$. Actually, $\Gamma_2 = \gamma$ and for $N \geq 3$, $\Gamma_N \in \mathbb{Z}[\gamma] \cap \mathbb{Z}[t^{\pm 2}] = \mathbb{Z}[\gamma^2]$ since $\varphi(N)$ is even. Define also

$\rho_N = t^{-\varphi(N)/2}\phi_N(t) = t^{\varphi(N)/2}\phi_N(t^{-1}) \in \mathbb{Z}[\gamma]$ for $N \geq 3$. From above, for $n \geq 2$, $\gamma_n = t^{1-n}(t^{2n}-1)/(t^2-1)$ factorizes in $\mathbb{Z}[t, t^{-1}]$ as

$$\gamma_n = \prod_{\substack{N|n\\N\geq 2}} \Gamma_N, \qquad \Gamma_N = \begin{cases} \rho_{2N} & \text{if } N \text{ is even} \\ \rho_N\rho_{2N} & \text{if } N \text{ is odd.}\end{cases} \tag{2.11.3}$$

From the factorization $\phi_N = \prod_{\substack{0<k<N\\ \gcd(k,N)=1}} \left(t - e^{\frac{2\pi k i}{N}}\right)$ in $\mathbb{C}[t, t^{-1}]$, we get

$$\rho_N = \prod_{\substack{0<k<\frac{N}{2}\\ \gcd(k,N)=1}} \left(\gamma - 2\cos\frac{2k\pi}{N}\right), \qquad \gamma_n = \prod_{k=1}^{n-1}\left(\gamma - 2\cos\frac{k\pi}{n}\right) \tag{2.11.4}$$

$$\Gamma_N = \prod_{\substack{0<k<N\\ \gcd(k,N)=1}} \left(\gamma - 2\cos\frac{k\pi}{N}\right) = \prod_{\substack{0<k<N/2\\ \gcd(k,N)=1}} \left(\gamma^2 - 4\cos^2\frac{k\pi}{N}\right) \tag{2.11.5}$$

in $\mathbb{R}[\gamma]$ for $n \geq 2$, $N \geq 3$. In particular, for $N \geq 3$, ρ_N is the minimum polynomial of $2\cos\frac{2\pi}{N}$ over $\mathbb{Q}$ and it is therefore irreducible in $\mathbb{Z}[\gamma]$. It follows that (2.11.3) gives the factorization of Γ_N into irreducibles in $\mathbb{Z}[\gamma]$. For odd N, $\phi_N(-t) = \phi_{2N}(t) \neq \pm\phi_N(t)$ so $\rho_N(-\gamma) \neq \rho_N(\gamma)$ which implies that $\rho_N, \rho_{2N} \notin \mathbb{Z}[\gamma^2]$. Hence for any $N \geq 3$, Γ_N is irreducible in $\mathbb{Z}[\gamma^2]$; in fact, it is the minimal polynomial of $4\cos^2\frac{\pi}{N}$ (as a polynomial in the variable γ^2 over $\mathbb{Z}$). Note $\Gamma_N \in \mathbb{Z}[xy]$ for $N \geq 3$ since $\gamma^2 = xy$.

Lemma. *Define elements of* $\mathbb{Z}[x,y]$ *by* $C_2 = \frac{\sqrt{x}}{\sqrt{y}}\gamma_2 = x$, $D_2 = \frac{\sqrt{y}}{\sqrt{x}}\gamma_2 = y$ *and* $C_N = D_N = \Gamma_N$ *for* $N \geq 3$. *Then the family* $\{C_2, D_2\} \cup \{C_N = D_N = \Gamma_N\}_{N\geq 3}$ *is a family of pairwise non-associate, irreducible elements of the unique factorization domain* $\mathbb{Z}[x,y]$ *and for any* $n \geq 2$, $c_n = \prod_{N|n, N>1} C_N$ *(resp.,* $d_n = \prod_{N|n,N>1} D_N$*) gives the factorization of* c_n *(resp.,* d_n*) into irreducibles in* $\mathbb{Z}[x,y]$.

Proof. The factorizations of c_n and d_n are immediate from (2.10.3) and (2.11.3). Now fix $N \geq 3$. It is easy to see Γ_N is irreducible in $\mathbb{Z}[x,y]$. (Any irreducible factor of Γ_N in $\mathbb{Q}[x,y]$ has non-zero constant term. Let S be the (finite) set of irreducible factors of Γ_N in $\mathbb{Q}[x,y]$ with constant term 1. Since $\Gamma_N \in \mathbb{Q}[xy]$, the map $\lambda \mapsto \big(F(x,y) \mapsto F(\lambda x, \lambda^{-1}y)\big)$ for $\lambda \in \mathbb{Q}^\bullet$ and F in S determines a group homomorphism from the multiplicative group $\mathbb{Q}^\bullet$ of $\mathbb{Q}$ to the symmetric group on S, with kernel K, say. For any $F = F(x,y) \in S$, we have $F(\lambda x, \lambda^{-1}y) = F(x,y)$ for all (infinitely many) $\lambda \in K$, and it readily follows that $F \in \mathbb{Q}[xy]$. Since Γ_N is irreducible in $\mathbb{Z}[\gamma^2]$ from above, $F(x,y)$ must be associate to Γ_N in $\mathbb{Q}[\gamma^2]$. This implies that Γ_N is irreducible in $\mathbb{Z}[x,y]$.) The Lemma follows readily. □

Remarks. In $\mathbb{Z}[v^{\pm 1}, w^{\pm 1}]$, the elements $C_2 = v + w^{-1}$, $D_2 = v^{-1} + w$ are associate, $C_N = D_N = (vw)^{-\varphi(N)/2}\phi_N(vw)$ is irreducible for $N \geq 3$, and the elements $\{C_N\}_{N\geq 2}$ are pairwise non-associate. We have

$$(C_N)_{N=2}^6 = (x, xy-1, xy-2, x^2y^2 - 3xy + 1, xy - 3). \tag{2.11.6}$$

2.12. For $z \in \{c, d, \gamma\}$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, define "binomial coefficients" in F by

$$\begin{bmatrix} m \\ n \end{bmatrix}_z := \frac{z_m z_{m-1}\cdots z_{m-n+1}}{z_n z_{n-1}\cdots z_1}.$$

It is well known that for $z=\gamma$ (resp., $z=c,d$) this element is in $\mathbb{Z}[t,t^{-1}]$ (resp., $\mathbb{Z}[v^{\pm 1},w^{\pm 1}]$); indeed, it is just a one-parameter (resp., two parameter) Gaussian binomial coefficient.

Lemma. *For any $m\in\mathbb{Z}$ and $n\in\mathbb{N}$, $\begin{bmatrix} m\\ n\end{bmatrix}_z$ is in $\mathbb{Z}[x,y]$ (resp., $\mathbb{Z}[\gamma]$) if $z=c$ or $z=d$ (resp., if $z=\gamma$).*

Proof. This can be deduced from the above-mentioned relation with Gaussian binomial coefficients, but follows readily from the next three identities as well:

$$\begin{bmatrix} m\\ n\end{bmatrix}_z = (-1)^n \begin{bmatrix} -m+n-1\\ n\end{bmatrix}_z \qquad \text{for } m<0, \tag{2.12.1}$$

$$\begin{bmatrix} m\\ n\end{bmatrix}_z = 0, \qquad \text{if } 0\le m<n \tag{2.12.2}$$

$$\begin{bmatrix} m\\ n\end{bmatrix}_z = \prod_{N\ge 2}(Z_N)^{\epsilon(N)} \qquad \text{if } 0\le n\le m \tag{2.12.3}$$

where $\epsilon(N)=\epsilon_{m,n}(N):=\lfloor \frac{m}{N}\rfloor - \lfloor\frac{m-n}{N}\rfloor - \lfloor\frac{n}{N}\rfloor \in\{0,1\}$ and where $Z=C$, D or Γ according as $z=c$, d or γ. Here, the third identity follows using Lemma 2.11 and (2.11.3) and for $m\in\mathbb{R}$, $\lfloor m\rfloor := \max\{n\in\mathbb{Z}\mid n\le m\}$. □

2.13. We shall describe below a process which converts a polynomial identity in $\mathbb{Z}[t,t^{-1}]$ involving t and Gaussian binomial coefficients $\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]_\gamma$ to an equivalent series of identities in $\mathbb{Z}[x,y]$. Observe first that, using (2.10.3),

$$\begin{bmatrix} m\\ n\end{bmatrix}_\gamma = \frac{\sqrt{y}}{\sqrt{x}}\begin{bmatrix} m\\ n\end{bmatrix}_c = \frac{\sqrt{x}}{\sqrt{y}}\begin{bmatrix} m\\ n\end{bmatrix}_d \qquad \text{if } n \text{ and } m-n \text{ are both odd} \tag{2.13.1}$$

$$\begin{bmatrix} m\\ n\end{bmatrix}_\gamma = \begin{bmatrix} m\\ n\end{bmatrix}_c = \begin{bmatrix} m\\ n\end{bmatrix}_d \qquad \text{if } n \text{ or } m-n \text{ is even.} \tag{2.13.2}$$

We endow the ring $\mathbb{Z}[t,t^{-1}]$ with a $\mathbb{Z}/2\mathbb{Z}$-grading with $\deg(t^n)=\overline{n}$ where $n\mapsto\overline{n}\colon\mathbb{Z}\to\mathbb{Z}_2$ is the canonical epimorphism. Also give the subring $\mathbb{Z}[x^{\pm 1/2},y^{\pm 1/2}]$ of F a $\frac{1}{2}\mathbb{Z}$-grading with $\deg(x^{m/2}y^{n/2})=\frac{m}{2}-\frac{n}{2}$. The latter grading restricts to a $\mathbb{Z}$-grading of the subring $\mathbb{Z}[x,y]$. We write $\deg(a)=m$ if a lies in the m-th homogeneous component of one of these rings (note the abuse of notation in the case of $\deg(0)$). Then we have

$$\deg(\gamma_k)=\overline{k-1}, \qquad \deg\left(\begin{bmatrix} m\\ n\end{bmatrix}_\gamma\right) = \begin{cases}\overline{1} & \text{if } m-n \text{ and } n \text{ are odd}\\ \overline{0} & \text{otherwise}\end{cases} \tag{2.13.3}$$

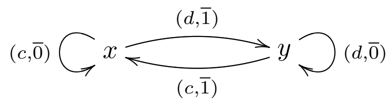
$$\deg(c_k) = -\deg(d_k) = \begin{cases} 1 & \text{if } k \text{ is even}\\ 0 & \text{otherwise}\end{cases} \tag{2.13.4}$$

$$\deg\left(\begin{bmatrix} m\\ n\end{bmatrix}_c\right) = -\deg\left(\begin{bmatrix} m\\ n\end{bmatrix}_d\right) = \begin{cases} 1 & \text{if } m-n \text{ and } n \text{ are odd}\\ 0 & \text{otherwise.}\end{cases} \tag{2.13.5}$$

Fix an integer k and consider the following elements of F:

$$T:=t^m\prod_{i=1}^r \begin{bmatrix} m_i\\ n_i\end{bmatrix}_\gamma, \quad T'_k := \gamma_{m+k}\prod_{i=1}^r \begin{bmatrix} m_i\\ n_i\end{bmatrix}_\gamma, \quad T_k := Y_{m+k}\prod_{i=1}^r \begin{bmatrix} m_i\\ n_i\end{bmatrix}_{X_i} \tag{2.13.6}$$

where $m \in \mathbb{Z}$ and where $Y = X_0, X_1, \dots, X_r \in \{c, d\}$ are chosen in the following way. Set $(m_0, n_0) = (m + k, 1)$. Choose the unique directed path in the graph



which begins with the vertex y such that the labels of the edges along the path are $(X_0, \epsilon_0), (X_1, \epsilon_1), \dots, (X_r, \epsilon_r)$ successively where $\epsilon_i = \overline{n_i(m_i - n_i)}$ in $\mathbb{Z}_2$.

One easily checks that $\eta = \deg(T_k) \in \{0, 1\}$. Then

$$\overline{\eta} = \deg(T'_k) = \deg(t^{k-1}T), \qquad T_k = \left(\frac{\sqrt{x}}{\sqrt{y}}\right)^{\eta} T'_k. \tag{2.13.7}$$

Since m is uniquely determined by T if $T \neq 0$, it follows that T_k depends only on T and k.

Proposition. *Fix $k \in \mathbb{Z}$. For each T as above, fix a choice of T_k as above. Let n_T be integers, all but finitely many of which are zero. Set $E := \sum_T n_T T \in \mathbb{Z}[t, t^{-1}]$ and $E_k := \sum_T n_T T_k \in \mathbb{Z}[x, y]$. Assume that E is homogeneous i.e. there is $\mu \in \mathbb{Z}/2\mathbb{Z}$ such that $n_T = 0$ if $\deg(T) \neq \mu$.*

(a) *If $E = 0$ in $\mathbb{Z}[t, t^{-1}]$, then $E_k = 0$ in $\mathbb{Z}[x, y]$.*

(b) *Fix an integer $l \geq 2$. Suppose that for every ring homomorphism*
$$\alpha \colon \mathbb{Z}[t, t^{-1}] \to A$$
such that A is an integral domain, $\alpha(t^{2l}) = 1$ and $\alpha(t^{2l'}) \neq 1$ for every integer l' satisfying $1 \leq l' < l$, we have $\alpha(E) = 0$. Then for every ring homomorphism $\beta \colon \mathbb{Z}[x, y] \to B$ such that B is an integral domain, $\beta(c_l) = 0$ and $\beta(c_{l'}) \neq 0$ for every integer l' satisfying $2 \leq l' < l$, we have $\beta(E_k) = 0$.

Remarks. (1) By (2.11.3) and Lemma 2.11, the intersection of the kernels of all the homomorphisms α (resp., β) as in (b) is the principal ideal of $\mathbb{Z}[t, t^{-1}]$ (resp., $\mathbb{Z}[x, y]$) generated by the homogeneous element Γ_l (resp., C_l).

(2) The hypothesis in the Lemma that E is homogeneous is not necessary (this is clear for (a) and follows using (1) for (b)).

(3) For $z \in \{c, d\}$, we have $f\left(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_z\right) = \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_\gamma$ where $f \colon \mathbb{Z}[x, y] \to \mathbb{Z}[\gamma]$ is the ring homomorphism determined by $f(x) = f(y) = \gamma$. Hence $f(E_k) = \sum_T n_T T'_k$. Using the identity $t\gamma_{m+k} - \gamma_{m+k-1} = t^{m+k}$, it follows that $E_k = 0$ and $E_{k-1} = 0$ together imply $E = 0$. A similar result holds in the situation of (b).

Proof. Recall the ring involution θ of $\mathbb{Z}[t, t^{-1}]$ given by $t \mapsto t^{-1}$. If $E = 0$, we have another homogeneous relation

$$E'_k := \frac{t^k E - \theta(t^k E)}{t - t^{-1}} = \sum_T n_T T'_k = 0$$

in $\mathbb{Z}[t, t^{-1}]$. By homogeneity of E, E_k is homogeneous; then (2.13.7) implies

$$E_k = E'_k \left(\frac{\sqrt{x}}{\sqrt{y}}\right)^{\deg E_k} = 0,$$

proving (a). For (b), it will suffice by (1) above to show that $\beta'(E) = 0$ where $\beta' \colon \mathbb{Z}[x, y] \to \mathbb{Z}[x, y]/\langle C_l \rangle$ is the canonical epimorphism. Choose an algebraically closed field K containing $\mathbb{Z}[x,y]/\langle C_l\rangle$ as a subring. We may construct in turn ring homomorphisms β'', τ, α making the following diagram commute

$$\begin{array}{ccccc}
\mathbb{Z}[x,y] & \hookrightarrow & \mathbb{Z}[\sqrt{x},\sqrt{y}\,] & \hookleftarrow & \mathbb{Z}[\gamma] \\
\downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\beta''} & \swarrow{\scriptstyle\tau} & \downarrow \\
\mathbb{Z}[x,y]/\langle C_l\rangle & \hookrightarrow & K & \xleftarrow{\alpha} & \mathbb{Z}[t,t^{-1}]
\end{array}$$

where the arrows $\hookrightarrow$ are the natural inclusions. We assert that $\alpha(t^{2l}) = 1$ and $\alpha(t^{2m}) \neq 1$ for $1 \leq m < l$, or equivalently in view of (2.10.5) and (2.11.3), that $\tau(\Gamma_l) = 0$, $\alpha(t-t^{-1}) \neq 0$, and $\tau(\Gamma_m) \neq 0$ for $2 \leq m < l$. Now $\beta''(\sqrt{y}) \neq 0$ since $\beta'(y) \neq 0$, so $\alpha(\Gamma_2) = \tau(\gamma) = \beta''(\sqrt{x}\sqrt{y}) = 0$ iff $\beta'(x) = 0$ i.e. iff $l = 2$. For $m \geq 3$, $\Gamma_m = C_m \in \mathbb{Z}[\gamma^2] \subseteq \mathbb{Z}[x,y]$ so $\tau(\Gamma_m) = \beta'(C_m)$. Finally, if $\alpha(t)$ equals 1 (resp., -1), then $\tau(\gamma)$ equals 2 (resp., -2) and using (2.10.4), (2.2.5) and Lemma 2.11, we see, since K has characteristic zero, that in either case $\tau(\Gamma_l) \neq 0$, contrary to the above. Now we have also $(\alpha\theta)(t^{2l}) = 1$ and $(\alpha\theta)(t^{2m}) \neq 1$ for $1 \leq m < l$. Using the hypothesis of (b), we get that $\alpha(t-t^{-1})\alpha(E'_k) = \alpha(t^kE) - (\alpha\theta)(t^kE) = 0$ and so $\tau(E'_k) = 0$, where $E'_k \in \mathbb{Z}[\gamma]$ is as in (a). Using (2.13.7) and homogeneity of E,

$$\beta'(E_k) = \tau(E'_k)\left(\frac{\beta''(\sqrt{x}\,)}{\beta''(\sqrt{y}\,)}\right)^{\deg E_k} = 0.$$

□

2.14. Let $\phi\colon \mathbb{Z}[x,y] \to A$ be a homomorphism of commutative rings. Many useful identities concerning the elements $\phi\big(\big[\begin{smallmatrix} n \\ m \end{smallmatrix}\big]_z\big)$ can be obtained from known identities on Gaussian binomial coefficients (see [32, 1.3, Ch 34] or [26] for instance) and Lemma 2.13. Part (a) of the following Lemma affords a non-trivial example.

Lemma. *Assume A is an integral domain and that $\phi(c_l) = 0$ for some non-zero $l \in \mathbb{Z}$. Fix l minimal and positive with $c_l = 0$ (note $l \geq 2$). For $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, write $n = ln_1 + n_0$ and $m = lm_1 + m_0$ where n_i, m_i are integers and $0 \leq n_0, m_0 < l$.*

(a) *Let $z = d$ if $m_0(n_0 - m_0)$ is an odd integer and $m(n-m)$ is even, and let $z = c$ otherwise. Also, set $k = n_0m_1 - n_1m_0 + (n_1+1)m_1l \in \mathbb{Z}$. Then*

$$\phi\left(\begin{bmatrix} n \\ m \end{bmatrix}_c\right) = \phi(z_{1+kl})\,\phi\left(\begin{bmatrix} n_0 \\ m_0 \end{bmatrix}_c\right)\binom{n_1}{m_1}$$

where $\binom{n_1}{m_1}$ is an ordinary binomial coefficient in $\mathbb{Z}$.

(b) *We have $\phi\big(\big[\begin{smallmatrix} n \\ m \end{smallmatrix}\big]_c\big) = 0$ in A iff $m_0 > n_0$ or $\binom{n_1}{m_1}$ is zero in A (i.e. divisible in $\mathbb{Z}$ by the characteristic of A); moreover, $\phi(z_{1+kl})$ is a unit in A.*

(c) *If $l = 2$ (i.e. $a = 0$), then $d_n = (-1)^{n_0+n_1+1}(bn_1)^{1-n_0}$ and*

$$\phi\left(\begin{bmatrix} n \\ m \end{bmatrix}_d\right) = (-1)^{m_1n_0 + m_0(n_0+n_1+1)}\big(b(n_1-m_1)\big)^{(1-n_0)m_0}\binom{n_1}{m_1}.$$

Proof. The identity ([32, Lemma 34.1.2.(c)])

$$\alpha\left(\begin{bmatrix} n \\ m \end{bmatrix}_\gamma\right) = \alpha\left(t^{kl}\begin{bmatrix} n_0 \\ m_0 \end{bmatrix}_\gamma\right)\binom{n_1}{m_1} \tag{2.14.1}$$

for homomorphisms α as in 2.13(b) gives (a). Part (b) follows using Lemma 2.9(i) and (2.12.3), since $\phi(C_f) \neq 0$ for $2 \leq f < l$. We shall not use (c), and leave its proof to the reader. □

2.15. In this subsection, let $\phi\colon \mathbb{Z}[x,y] \to A$ be a homomorphism of $\mathbb{Z}[x,y]$ into an integral domain A. Assume $\phi(c_l) = 0$ for some $l > 0$, and let $l \in \mathbb{N}$ be the smallest such integer. Let p be the characteristic of A.

Proposition. *Fix $n, m, r := n + m \in \mathbb{N}$. Define $n_i, m_i, r_i \in \mathbb{N}$ as follows. If $p = 0$, then for $N = n$, m or r, write $N = N_1 l + N_0$ with $N_i \in \mathbb{N}$ and $0 \leq N_0 < l$. If $p > 0$, set $p_0 = l$, $p_i = p$ for $i \geq 1$ and $P_i = \prod_{i=0}^{i-1} p_i$ for $i \geq 0$, so $P_0 = 1$; for $N = n$, m or r, write $N = \sum_{i \in \mathbb{N}} N_i P_i$ with $0 \leq N_i < p_i$.*

(a) *$\phi\left(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c\right) = 0$ iff $n_i > m_i$ for some i.*
(b) *$\phi\left(\left[\begin{smallmatrix} -m-1 \\ n \end{smallmatrix}\right]_c\right) = 0$ iff $m_i > r_i$ for some i.*

Remarks. We shall require the above expression $N = \sum_i N_i P_i$ again subsequently; we call it the l-modified p-adic expansion of N.

Proof. It is well known that an ordinary binomial coefficient $\binom{m}{n}$ in $\mathbb{Z}$ with $m, n \geq 0$ is divisible by a prime p (i.e. zero in $\mathbb{Z}/p\mathbb{Z}$) iff $\epsilon_{m,n}(p^i) > 0$ for some $i > 0$, where $\epsilon_{m,n}$ is as defined in 2.12. (This also follows inductively from Lemma 2.14(b) by considering $\phi\colon \mathbb{Z}[x,y] \to \mathbb{Z}/p\mathbb{Z}$ with $\phi(x) = \phi(y) = 2$). On the other hand, $\binom{m}{n}$ is zero in $\mathbb{Z}$ iff $n > m \geq 0$. Part (a) follows immediately from this and Lemma 2.14(b). Then (b) follows from (a) on applying ϕ to the identity

$$\left[\begin{smallmatrix} -m-1 \\ n \end{smallmatrix}\right]_c = (-1)^n \left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right]_c. \tag{2.15.1}$$

□

2.16. Many of the above identities involving c_n, d_n and binomial coefficents modelled on these sequences have (trivial) extensions to non-commutative rings. We briefly indicate the necessary changes. Below, we use the convention that for elements x_j of a non-commutative ring R, $\prod_{j=p}^{q} x_j := x_p x_{p+1} \cdots x_q$ for integers $p \leq q$.

First, one defines $c_n(a,b)$ and $d_n(a,b)$ for a, b in a (possibly non-commutative) unital ring R by exactly the same recurrence formula and initial condition as in the commutative case, so for instance

$$(c_n)_{n=0}^{6} = (0, 1, a, ba-1, aba-2a, baba-3ba+1, ababa-4aba+3a) \tag{2.16.1}$$

and $d_n(a,b) = c_n(b,a)$. Now define $C_n = C_n(a,b)$ and $D_n = D_n(a,b)$ as follows. Let $\langle ba \rangle$ (resp., $\langle ab \rangle$ denote the (commutative) subring of R generated by 1 and ba (resp., 1 and ab). Let $C_2 = a$, $D_2 = b$, and C_N (resp., D_N) for $N \geq 3$ denote the minimal polynomial of $4\cos^2 \frac{\pi}{N}$ evaluated at $ba \in \langle ba \rangle$ (resp., $ab \in \langle ab \rangle$). We have for $n \geq 2$ that $c_n = \prod_{N|n, N \geq 2} C_n$ (resp., $d_n = \prod_{N|n, N\geq 2} D_n$). Note that the order in which the factors are taken is actually irrelevant except that C_2 (resp., D_2) must come first if n is even. Similarly, one may define $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c \in \langle ba \rangle \cup a\langle ba \rangle$ (resp., $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_d \in \langle ab \rangle \cup b\langle ab \rangle$) by the three identities (2.12.1)–(2.12.3) with $z = c$, $Z = C$ (resp., $z = d$, $Z = D$).

In particular, take R to be the free ring $\mathbb{Z}\langle a, b \rangle$ on two non-commuting variables a, b. Suppose given a relation $E = \sum_T n_T T = 0$ in $\mathbb{Z}[t, t^{-1}]$ (resp., $\mathbb{Z}[t, t^{-1}]/\langle \phi_l(t^2) \rangle$ with $l \geq 2$) as in Proposition 2.13, where T runs over terms as in (2.13.6) and $n_T \in \mathbb{Z}$. Define $T_k \in \mathbb{Z}\langle a, b \rangle$ formally as in the third equation in (2.13.6), regarding terms c, d and the binomial coefficients there as being given by their non-commutative versions defined above (here, the convention on ordering of terms of the product is essential). Then one checks that $T_k \in \langle ba \rangle \cup a\langle ba \rangle$ and it follows readily from Proposition 2.13 that $E_k := \sum_T n_T T_k = 0$ in $\mathbb{Z}\langle a, b \rangle$ (resp., $\mathbb{Z}\langle a, b \rangle / \langle C_l \rangle$).

2.17. The c_n again arise as root coefficients (for reflection representations of dihedral groups over possibly non-commutative coefficent rings). The author does not not know of any interpretation of the corresponding "binomial coefficents" defined above, for instance analogous to their interpretation in Proposition 2.25 in the commutative case. We do however mention the following observation.

Let $\mathcal{H}$ denote the generic Iwahori-Hecke algebra ([27]) of an infinite dihedral Coxeter system (W, S) with $S = \{r, s\}$. For $w \in W$, let C_w denote the corresponding Kazhdan-Lusztig basis element of $\mathcal{H}$ defined in loc cit. Fix an integer $n \geq 1$ and let $w := \big((sr)_n\big)^{-1} \in W$. Then

$$C_w = c_n(C_r, C_s)C_s$$

(as follows readily on comparing the recurrence formulae (2.2.1) with [27, (2.3)(a)] and using the well-known fact that the (non-zero) Kazhdan-Lusztig polynomials for dihedral groups are all equal to 1).

2.18. **Valuations of Binomial Coefficients.** In 2.18-2.23, we shall establish a formula for the valuation of an element $\phi\big(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c\big)$, where $\phi\colon \mathbb{Z}[x, y] \to A$ is a homomorphism into a valuation ring A. This result is not used subsequently in this paper. First, we collect some additional facts concerning γ_n and Γ_n (cf [26] for the first few of these).

The Chebyshev functions $T_n(x) := \cos(n \arccos x)$ and $U_n(x) := \sin(n \arccos x)$ of the real variable x are defined for $n \in \mathbb{Z}$ and provide for $n > 0$ two linearly independent solutions of Chebyshev's differential equation $(1-x^2)y'' - 2xy' + n^2 y = 0$. We now regard x as an indeterminate and $T(x) \in \mathbb{R}[x]$, $U(x) \in \mathbb{R}[x, \sqrt{1-x^2}]$. It is well known (and easily checked) that $T_n(x)$ and $U_n(x)$ both satisfy the recurrence formula $f_{n+1} - 2xf_n + f_{n-1} = 0$ which is also satisfied by $\gamma_n(2x)$; comparing initial conditions,

$$\gamma_n(x) = U_n\Big(\frac{x}{2}\Big)/U_1\Big(\frac{x}{2}\Big), \qquad \gamma_{n+1}(x) - \gamma_{n-1}(x) = 2T_n\Big(\frac{x}{2}\Big) \tag{2.18.1}$$

for $n \in \mathbb{Z}$ where $U_1(\frac{x}{2}) = \sqrt{1 - \frac{x^2}{4}}$. We have the generating function formulae

$$\frac{1}{1 - z\gamma + z^2} = \sum_{n=0}^{\infty} \gamma_{n+1} z^n, \qquad \frac{1 - z^2}{1 - z\gamma + z^2} = 1 + \sum_{n=1}^{\infty} (\gamma_{n+1} - \gamma_{n-1}) z^n \tag{2.18.2}$$

in the power series ring $(\mathbb{Z}[\gamma])[[z]]$ and formulae

$$\gamma_n = -\gamma_{-n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-1-j}{j} \gamma^{n-1-2j} \quad \text{for } n \geq 0, \tag{2.18.3}$$

(2.18.4)

$$\gamma_{n+1} - \gamma_{n-1} = t^n + t^{-n} = \gamma^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n}{j} \binom{n-1-j}{j-1} \gamma^{n-2j} \quad \text{for } n \geq 1$$

in $\mathbb{Z}[\gamma] \subseteq \mathbb{Z}[t, t^{-1}]$. From (2.11.4), it follows that for $N \geq 3$, Γ_N is a monic polynomial of degree $\varphi(N)/2$ in $\mathbb{Z}[\gamma^2]$, with successive coefficients of alternating signs. From above, the constant term of γ_{2n+1} is $(-1)^n$ and the coefficient of γ in

γ_{2n} is $(-1)^{n+1}n$, for $n \geq 0$. Using (2.11.3), it follows that for $N \geq 3$,

(2.18.5) $$\Gamma_N(0) = \begin{cases} (-1)^{\varphi(N)/2}p & \text{if } N = 2p^m,\ p \text{ prime},\ m \geq 1 \\ (-1)^{\varphi(N)/2} & \text{otherwise.} \end{cases}$$

From (2.11.3) and Lemma 2.11,

(2.18.6) $$\Gamma_n = \prod_{N|n, N>1} \gamma_N^{\mu(n/N)}, \quad C_n = \prod_{N|n, N>1} c_N^{\mu(n/N)} \qquad n \geq 2$$

where μ is the Möbius function. By definition, $\Gamma_n(t+t^{-1}) = t^{-\varphi(n)}\phi_n(t^2)$ for $n \geq 2$. From (2.11.1), this gives

(2.18.7) $$\Gamma_{2N} = (-1)^{\varphi(N)/2}\Gamma_N\Big(\sqrt{4-\gamma^2}\Big), \qquad \text{for odd } N > 1$$

i.e. if $\Gamma_N = f_N(\gamma^2)$, then $\Gamma_{2N} = (-1)^{\varphi(N)/2} f_N(4-\gamma^2)$. Now fix an integer $N \geq 1$ and a prime integer $p > 0$. Similarly by (2.11.2), recalling $t^p + t^{-p} = \gamma_{p+1} - \gamma_{p-1}$,

(2.18.8) $$\Gamma_{Np} = \begin{cases} \gamma_p & \text{if } N = 1 \\ \Gamma_N(\gamma_{p+1} - \gamma_{p-1}) & \text{if } p|N \\ \frac{\Gamma_N(\gamma_{p+1}-\gamma_{p-1})}{\Gamma_N(\gamma)} & \text{otherwise.} \end{cases}$$

A special case of this is the formula $\Gamma_{2^{n+1}} = \Gamma_{2^n}(\gamma^2 - 2)$ for $n \geq 1$, which implies

(2.18.9) $$\Gamma_{2^n}\big(\sqrt{4-\gamma^2}\big) = (-1)^{\varphi(2^n)/2}\Gamma_{2^n}(\gamma) \qquad \text{for } n \geq 2.$$

Another special case (taking $N = 2$) is

(2.18.10) $$\gamma_{p+1} - \gamma_{p-1} = \begin{cases} \gamma\Gamma_{2p} & \text{if } p \text{ is odd} \\ \Gamma_{2p} & \text{if } p = 2\ . \end{cases}$$

Now $\gamma_{p+1} - \gamma_{p-1} \equiv (t+t^{-1})^p = \gamma^p \pmod p$, so it follows by induction that

(2.18.11) $$\Gamma_{2p^m} \equiv \gamma^{\varphi(2p^m)} \pmod p \qquad \text{for prime } p \text{ and } m \geq 1.$$

Thus, for $m \geq 1$, Γ_{2p^m} is a monic integral polynomial of degree $\varphi(2p^m)/2$ in the variable γ^2, with constant term $\pm p$ and all coefficients other than the leading coefficient divisible by p (it is an Eisenstein polynomial for p).

2.19. We give specializations of some of the above formulae for Γ_N under a ring homomorphism $\pi\colon \mathbb{Z}[\gamma] \to S$, where S is commutative ring of characteristic p. Let $F_S\colon S \to S$ be the Frobenius homomorphism of S, given by $\lambda \mapsto \lambda^p$ for $\lambda \in S$. Since $\pi(\gamma_{p+1} - \gamma_{p-1}) = \pi(\gamma^p) = (F_S\pi)(\gamma)$ and $\pi(\Gamma_N(\gamma^p)) = (F_S\pi)(\Gamma_N)$ for any $N > 0$, we get that for $M, n \in \mathbb{Z}_{>0}$ with $\gcd(M,p) = 1$,

(2.19.1) $$\pi(\Gamma_{Mp^n}) = \begin{cases} \big(F_S^{n-1}\pi\big)(\Gamma_p) & \text{if } M = 1 \\ \big(F_S^{n-1}\pi\big)(\Gamma_M^{p-1}) & \text{if } M > 1. \end{cases}$$

If S is an integral domain, we see that for an integer $N \geq 2$,

(2.19.2) $$\pi(\Gamma_N) = 0 \quad \text{iff} \quad \pi(\Gamma_{pN}) = 0.$$

If S is a finite field with $q = p^m$ elements, we also get

(2.19.3) $$\pi(\Gamma_{Nq}) = \pi(\Gamma_N) \quad \text{if} \quad p|N.$$

2.20. In this subsection, fix an integer $l \geq 2$. We suppose $A = \mathbb{Z}[x,y]/\langle C_l \rangle$ and that $\phi\colon \mathbb{Z}[x,y] \to A$ is the canonical epimorphism, which we write as $f \mapsto \overline{f}$ for $f \in \mathbb{Z}[x,y]$. We write $f \sim g$ to indicate that elements f, g of the integral domain A are associate.

Lemma. *Let $m \geq 2$ be an integer.*

(a) *If $m = q^n l$ with q a positive prime and $n \geq 1$, then $\overline{C}_m \sim q$ in A.*
(b) *$\overline{C}_m$ is a non-unit of $\overline{A}$ iff $m = q^n l$ for some prime integer q and some $n \in \mathbb{Z}$.*

Proof. First, we show that if $m \nmid l$ and $l \nmid m$, then $\overline{C}_m$ is a unit. Write $\gcd(m,l) = g$. By Lemma 2.9(a), $\overline{c}_m A = \overline{c}_g A \neq 0$. Hence $A \prod_{d\colon 2 \leq d | m} \overline{C_d} = A \prod_{d\colon 2 \leq d | g} \overline{C_d} \neq 0$ by Lemma 2.11, giving $\prod_{d\colon d|m, d \nmid g} \overline{C}_d \sim 1$ and so $\overline{C}_m \sim 1$. Now suppose that $m = nl$ with $n \in \mathbb{Z}_{>1}$. Using Lemma 2.14 and (2.12.3), we get

$$n \sim \phi\left(\begin{bmatrix} nl \\ l \end{bmatrix}_c \right) \sim \prod_{N \geq 2} (\overline{C}_N)^{\epsilon_{nl,l}(N)} \sim \prod_{j : 2 \leq j | n} \overline{C}_{jl}$$

since $\overline{C}_N$ is unit unless $N \mid l$ or $l \mid N$, in which case $\epsilon_{nl,l}(N) \neq 0$ iff $N = jl$ with $2 \leq j \mid n$. It follows readily that $\overline{C}_{nl} \sim 1$ unless n is a positive power of a prime q, in which case $C_{nl} \sim q$. Now $\overline{C_m}$ is a unit in A iff $A/\langle \overline{C}_m \rangle = \mathbb{Z}[x,y]/\langle C_l, C_m \rangle = 0$. From what we've seen already, it will therefore suffice to prove (b) under the assumption that $l|m$ (interchanging l and m if necessary). This follows from the above since for $k \in \mathbb{Z}$, k is invertible in A iff k is invertible in $\mathbb{Z}$. ☐

2.21. Let $l \in \mathbb{Z}_{\geq 2}$. Write $C_n = C_n(x,y)$ to indicate dependence on x and y. Set $(a', b') := (c_{l+1} - c_{l-1}, d_{l+1} - d_{l-1}) \in \mathbb{Z}[x,y]^2$ and define $c'_n = c_n(a', b')$, $C'_n = C_n(a', b')$ in $\mathbb{Z}[x,y]$ for $n \geq 2$. Thus, by (2.8.7) and Lemma 2.11,

(2.21.1) $$c'_n = \prod_{N|n, N \geq 2} C'_N, \qquad c_n = \prod_{N|n, N \geq 2} C_N, \qquad c_{nl} = c_l c'_n$$

and similarly for d_n. From this, it follows that for $n \geq 2$, the factorization of C'_n into irreducibles in $\mathbb{Z}[x,y]$ is given by

(2.21.2)
$$C'_n = \prod_{j \in I_{n,l}} C_j, \qquad I_{n,l} := \{\, j \in \mathbb{N} \mid j|nl, j \nmid ml \text{ for } m|n \text{ with } 1 \leq m < n \,\}.$$

2.22. Let L be a field and $\nu\colon L \to \mathbb{R} \cup \{\infty\}$ be an additive valuation i.e. a map satisfying

(i) $\nu(x) = \infty$ iff $x = 0$
(ii) $\nu(xy) = \nu(x) + \nu(y)$
(iii) $\nu(x + y) \geq \min(\nu(x), \nu(y))$

for $x, y \in L$ (we allow trivial ν i.e. $\nu(x) = 0$ for all $x \neq 0$). Let A be the valuation ring of L (i.e. $A = \{x \in L \mid \nu(x) \geq 0\}$) and $\mathfrak{m} = \{x \in L \mid \nu(x) > 0\}$ be the maximal ideal of the (local) ring A. Let $\phi\colon \mathbb{Z}[x,y] \to A$ be a ring homomorphism, and $\pi\colon A \to A/\mathfrak{m}$ be the natural epimorphism.

If $\nu(c_j) = 0$ (i.e. $c_j \notin \mathfrak{m}$) for all $j \geq 2$, then for any integers m and n, $(\nu\phi)\left(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c\right)$ is zero unless $0 \leq m < n$, in which case it is ∞. We assume henceforward that $\nu(c_j) > 0$ for some $j \geq 2$.

Lemma. *Set $p := \operatorname{char}(A/\mathfrak{m})$ and $l := \min\{\, j \in \mathbb{Z}_{>0} \mid \nu(c_j) > 0 \,\} \geq 2$. Then*

(a) *For* $n \geq 2$, $\nu(\phi(C_n)) \neq 0$ *iff* $n = lp^m > 0$ *for some* $m \in \mathbb{N}$.
(b) *If* $p|l$ *then* $l = p$ *or* $l = 2p$.
(c) *Suppose that* $p > 0$ *and* $\mathrm{char}(A) = p$. *Then* $\nu(\phi(C_{lp^n}) = p^{n-1}\nu(\phi(C_{lp})$ *for all* $n \geq 1$.
(d) *Suppose that* $p > 0$ *and* $\mathrm{char}(A) = 0$. *One may chose an integer* $M \geq 2$ *so that* $\varphi(p^M)(\nu\phi)\big(c_l d_l(4-xy)\big) > 2\nu(p)$. *Then* $\nu(C_{lp^n}) = \nu(p)$ *for all* $n \geq M$.
(e) *For integers* $0 \leq m \leq n$,

$$\nu\left(\phi\left(\begin{bmatrix} m \\ n \end{bmatrix}_c\right)\right) = \sum_{j\in\mathbb{N}:lp^j>0} \epsilon_{m,n}(lp^j)\nu(\phi(C_{lp^j}))$$

where $\epsilon(N) = \epsilon_{m,n}(N) := \lfloor \frac{m}{N} \rfloor - \lfloor \frac{m-n}{N} \rfloor - \lfloor \frac{n}{N} \rfloor \in \{0,1\}$ *and where we interpret* $0 \cdot \infty$ *as* 0.

Proof. For $m \in \mathbb{Z}$, $\nu(\phi(C_m)) > 0$ iff $\pi\phi(C_m) = 0$. Now $\pi\phi$ factors as a composite $\mathbb{Z}[p,q] \to \mathbb{Z}[p,q]/\langle C_l \rangle \to A/\mathfrak{m}$. Suppose $m \neq l$ with $(\pi\phi)(C_m) \neq 0$. By 2.20(b), we have $m = lq^n$ for some prime q and $n \neq 0$. By choice of l, $n \geq 1$. By 2.20(a), we get $0 = (\pi\phi)(C_m) = q$, so $p = q > 0$. Suppose on the other hand that $p > 0$. By 2.20(a), $(\pi\phi)(C_{p^n l}) = p = 0$ for $n > 0$. This proves (a). If $p > 0$ and $m > 2$, then $(\pi\phi)(C_{pm}) = 0$ iff $(\pi\phi)(C_m) = 0$ by (2.19.2), since $C_m = \Gamma_m$ for $m > 2$. This proves (b). If $\mathrm{char}(A) = p > 0$, then we have similarly that $\phi(C_{lp^n}) = \phi(C_{lp})^{p^{n-1}}$ for $n \geq 1$ by (2.19.1), since $C_{lp} = \Gamma_{lp}$ and $p|lp$. This proves (c).

Suppose now that $p > 0$ and $\mathrm{char}(A) = 0$. Since $\nu(c_l) > 0$ and $\nu(p) \neq \infty$, we may certainly choose the integer M as in (d). We shall now use the notation of 2.21. Observe that in (2.21.2), we have $\nu(C_j) = 0$ if $j \in I_{n,l}$ and $j \neq nl$, so we get that $(\nu\phi)(C_{nl}) = (\nu\phi)(C'_n)$. For $N \geq 3$, write $\Gamma_N = f_N(\gamma^2)$ for some polynomial $f_N \in \mathbb{Z}[\gamma]$. From (2.18.7), (2.18.9) and Lemma 2.6(b), it follows that for $n \geq 2$

$$C'_{p^n} = f_{p^n}(a'b') = f_{p^n}\big(4 - c_l d_l(4 - xy)\big) = \pm g_n\big(c_l d_l(4 - xy)\big) \tag{2.22.1}$$

where

$$g_n(\gamma) := \begin{cases} f_{2p^n}(\gamma) & \text{if } p \neq 2 \\ f_{p^n}(\gamma) & \text{if } p = 2. \end{cases} \tag{2.22.2}$$

Now $g_n(\gamma) \in \mathbb{Z}[\gamma]$ is a monic Eisenstein polynomial for the prime p, of degree $\varphi(p^n)/2$ and with constant term $\pm p$. Writing $\mu = (\nu\phi)\big(c_l d_l(4 - xy)\big) > 0$, the non-Archimedean property of the valuation gives for $n \geq 2$

$$(\nu\phi)(C_{lp^n}) = (\nu\phi)(C'_{p^n}) \begin{cases} = \frac{1}{2}\varphi(p^n)\,\mu & \text{if } \frac{1}{2}\varphi(p^n)\,\mu < \nu(p) \\ \geq \nu(p) & \text{if } \frac{1}{2}\varphi(p^n)\,\mu = \nu(p) \\ = \nu(p) & \text{if } \frac{1}{2}\varphi(p^n)\,\mu > \nu(p). \end{cases} \tag{2.22.3}$$

The last case gives (d). Part (e) follows by taking valuations of both sides in the following formula, which follows from (2.12.3):

$$\phi\left(\begin{bmatrix} m \\ n \end{bmatrix}_c\right) = \prod_{N\geq 2} (\phi(C_N))^{\epsilon_{m,n}(N)}. \tag{2.22.4}$$

□

2.23. Maintaining the assumptions of the last subsection, we immediately get the following.

Corollary. *Fix* $n, m, r := n + m \in \mathbb{N}$ *and for* $N = n, r, m$ *define the* l*-modified* p*-adic expansion* $N = \sum_i N_i P_i$ *as in* Proposition 2.15. *Then,*

(a) $(\nu\phi)\left(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c\right) = \sum_{i\colon n_i > m_i} \nu(\phi(C_{lp^i}))$ *if* $0 \leq n \leq m$ *and* $\phi\left(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c\right) = 0$ *if* $0 \leq m < n$.

(b) $(\nu\phi)\left(\left[\begin{smallmatrix} -m-1 \\ n \end{smallmatrix}\right]_c\right) = \sum_{i\colon m_i > r_i} \nu(\phi(C_{lp^i}))$.

Remarks. According to (a), for $0 \leq n \leq m$, $(\nu\phi)\left(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_c\right)$ is the weighted sum of the "carries" when m is added to $n - m$ using l-modified p-adic notation, with a carry in the i-th place ($i \geq 0$) being accorded weight $\nu(\phi(C_{lp^i}))$. Moreover, the sequence of weights becomes regular for sufficiently large i, by Lemma 2.22(c)–(d). This generalizes the well-known formula for p-adic valuations of ordinary binomial coefficients.

2.24. **BGG-Demazure operators.** In 2.3, we take $R = \mathbb{R}$, $v = 1$, and assume that $\Pi := \{A, B\}$, $\Pi^\vee = \{A^\vee, B^\vee\}$ afford sets of $\mathbb{R}$-linearly independent simple roots and coroots for a reduced root system of a reflection representation of a dihedral Coxeter system (W, S) (so $a \geq 0$, $b \geq 0$, $a = 0$ iff $b = 0$, $ab \in \{\, 4\cos^2 \frac{\pi}{m} \,\} \cup [4, \infty)$, and $a = b$ if $ab = 4\cos^2 \frac{\pi}{m}$ with m odd). We define $m \in \mathbb{Z}_{\geq 2}$ by $m = \infty$ if $ab \geq 4$, and $ab = 4\cos^2 \frac{\pi}{m}$ otherwise, so W has order $2m$. If m is infinite, every root in Φ_+ is uniquely of the form $\pm\alpha_p$ for some integer p; if m is finite, $\alpha_{j+m} = -\alpha_j$ and every root is equal to a unique one of $\alpha_0, \ldots, \alpha_{2m-1}$. We denote s_A as r and s_B as s. We recall that $\leq^T$ (resp., $\leq^\emptyset$) denotes reverse (resp., ordinary) Chevalley-Bruhat order on W with maximum (resp., minimum) element 1.

Lemma. *Fix* $I = \emptyset$ *or* $I = T$ *and denote* $\leq^I$, l^I *just as* $\leq$, l. *Consider* $x \leq y$ *in* W' *with* $l(y) - l(x) = 2n + 1$ *and choose* $p \in \mathbb{Z}$ *such that* $x = s_{\alpha_p} y$. *Then*

(a) *there exist* $\epsilon \in \{\pm 1\}$ *and* $q \in \mathbb{N}$ *such that* $\{\, t \in T \mid x \leq ty < y \,\} = \{\, s_{\alpha_j} \mid j \in X \,\}$ *where* $X = \{p, p + \epsilon, \ldots, p + n\epsilon\} \cup \{p - (q+1)\epsilon, p - (q+2)\epsilon, p - (q+n)\epsilon\}$.

(b) $\alpha_{p-(q+1)\epsilon}\alpha_{p-(q+2)\epsilon} \cdots \alpha_{p-(q+n)\epsilon} \equiv \pm c^I_{x,y} \alpha_{p+\epsilon}\alpha_{p+2\epsilon} \cdots \alpha_{p+n\epsilon} \pmod{\alpha_p}$ *in the ring* $\mathbb{R}[A, B]$.

Remarks. If $q = 0$, then in (a) one could replace ϵ by $-\epsilon$ and still have the same set X, while in (b) one would have $c_{x,y} = 1$ since $\alpha_{p+k\epsilon} \equiv -\alpha_{p-k\epsilon} \pmod{\alpha_p}$.

Proof. The first assertion is clear from inspection of Figure 2. For the second, we have $S^I_{x,y} = \pm \frac{c^I_{x,y}}{\prod_{j \in X} \alpha_j}$. From (1.9.3)–(1.9.5) and Figure 2, on the other hand, it follows by induction (we omit the detailed computations in the dihedral group) that $S^I_{x,y} \equiv \pm \frac{1}{\prod_{j=1}^n \alpha^2_{p+j\epsilon}} \pmod{\mathbb{R}[A, B]_{(\alpha_p)}}$ where $real[A, B]_{(\alpha_p)}$ is the localization of $\mathbb{R}[A, B]$ at its multiplicative subset generated by all roots not proportional to α_p. Equating these two expressions for $S^I_{x,y}$ and clearing denominators gives the congruence in (b). □

2.25. We continue to abbreviate $c_n = c_n(x, y)$ and $d_n = d_n(x, y)$, where x, y are generators of a polynomial ring $\mathbb{Z}[x, y]$, so $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_z \in \mathbb{Z}[x, y]$ for $z = c, d$. Let $\phi\colon \mathbb{Z}[x, y] \to \mathbb{R}$ be the ring homomorphism with $\phi(x) = a$, $\phi(y) = b$.

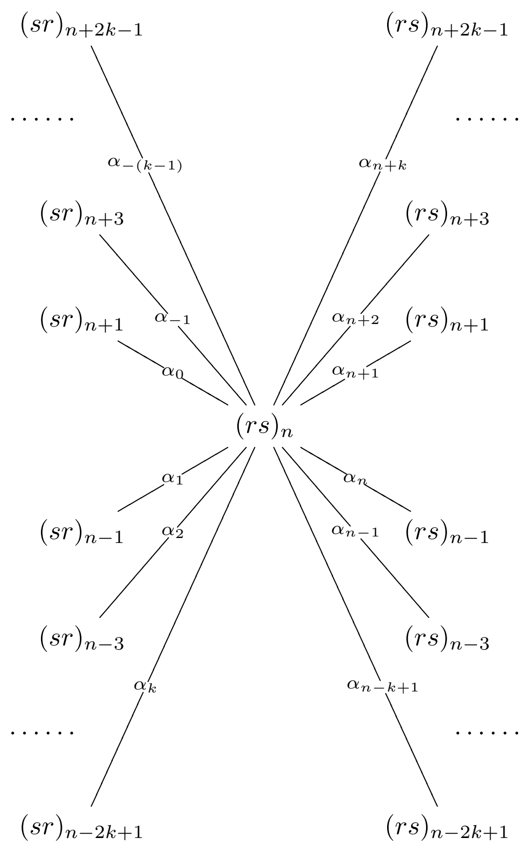


FIGURE 2. Diagram of edges incident with a vertex in the Bruhat graph of a dihedral group. A label of γ on an edge joining x, y means that γ is a (not necessarily positive) root with $y = s_\gamma x$.

Proposition. *The elements $c_{x,y} = c^T_{x,y}$ and $c^\emptyset_{x,y}$ in $\mathbb{Z}_{\geq 0}$ defined in* 1.6 *and* 1.20 *for $x, y \in W$, are given in the case of the dihedral group W of order $2m$ by the following formulae:*

$$c^T_{(rs)_n,(sr)_{n-k}} = \phi\left(\begin{bmatrix} n-1 \\ \lfloor\frac{k-1}{2}\rfloor\end{bmatrix}_c\right) \qquad \text{if } 0 \leq n-k < n < m+1$$

$$c^T_{(rs)_n,(rs)_{n-k}} = \phi\left(\begin{bmatrix} n-1 \\ \lfloor\frac{k}{2}\rfloor\end{bmatrix}_c\right) \qquad \text{if } 0 \leq n-k \leq n < m+1$$

$$c^\emptyset_{(sr)_n,(rs)_{n+k}} = (-1)^{\lfloor\frac{k-1}{2}\rfloor}\phi\left(\begin{bmatrix} -n-1 \\ \lfloor\frac{k-1}{2}\rfloor\end{bmatrix}_c\right) \qquad \text{if } 0 \leq n < n+k < m+1$$

$$c^\emptyset_{(sr)_n,(sr)_{n+k}} = (-1)^{\lfloor\frac{k}{2}\rfloor}\phi\left(\begin{bmatrix} -n-1 \\ \lfloor\frac{k}{2}\rfloor\end{bmatrix}_c\right) \qquad \text{if } 0 \leq n \leq n+k < m+1.$$

Remarks. There are similar formula obtained by symmetry (interchange $r \leftrightarrow s$ on the left hand sides and $c \leftrightarrow d$ on the right hand sides). The second pair of formulae (except for the sign and the inequalities) is obtained formally from the first pair on replacing T by $\emptyset$ and n by $-n$, recalling our notational convention $(rs)_{-n} = (sr)_n$.

Proof. By (1.5)(iii),

$$c^I_{y,x} = c^I_{y,sx} \quad \text{for } y \le x \text{ and } \gamma \in \Pi \text{ with } s_\gamma y >^I y \text{ and } s_\gamma x <^I x \tag{2.25.1}$$

since $\Phi^I_{y,x} = s_\gamma(\Phi^I_{y,s_\gamma x}) \cup \{\gamma\}$. Hence it is enough to prove the formulae for odd k. Since both sides of the identities are positive, it is enough to establish them up to sign. There are four cases; we shall treat only one of them in detail.

Case (1). For $k \ge 1$ with $n + 2k - 1 < m + 1$, the preceding Lemma and Figure 2 gives the following congruence in $\mathbb{R}[A, B]$:

$$c^T_{(sr)_{n+2k-1},(rs)_n} \alpha_0 \alpha_{-1} \cdots \alpha_{-(k-2)} \equiv \pm \alpha_{n+1} \alpha_{n+2} \cdots \alpha_{n+k-1} \pmod{\alpha_{-(k-1)}}.$$

From Lemma 2.7, we have $F_{-k,s}\, \alpha_{-(k-1)} + F_{s,-(k-1)}\, \alpha_{-k} + F_{-(k-1),-k}\, \alpha_s = 0$ and so since $F_{-(k-1),-k} = -1$, we get $\alpha_s \equiv F_{s,-(k-1)} \alpha_{-k} \pmod{\alpha_{-(k-1)}}$. In particular, α_{-k} and $\alpha_{-(k-1)}$ span V, so they form a basis for V. Substituting into the congruence above gives

$$\begin{aligned} c^T_{(sr)_{n+2k-1},(rs)_n} & F_{-(k-1),0} F_{-(k-1),-1} \cdots F_{-(k-1),-(k-2)} \alpha^{k-1}_{-k} \\ & \equiv \pm F_{-(k-1),n+1} F_{-(k-1),n+2} \cdots F_{-(k-1),n+k-1} \alpha^{k-1}_{-k} \pmod{\alpha_{-(k-1)}} \end{aligned}$$

and so

$$\begin{aligned} c^T_{(sr)_{n+2k-1},(rs)_n} &= \pm \frac{F_{-(k-1),n+1} F_{-(k-1),n+2} \cdots F_{-(k-1),n+k-1}}{F_{-(k-1),0} F_{-(k-1),-1} \cdots F_{-(k-1),-(k-2)}} \\ &= \pm \begin{cases} \frac{c_{n+k} \cdots c_{n+2k-2}}{c_1 \cdots c_{k-1}} & \text{for odd } k \\ \frac{d_{n+k} \cdots d_{n+2k-2}}{d_1 \cdots d_{k-1}} & \text{for even } k \end{cases} = \pm \phi \left(\begin{bmatrix} n + 2k - 2 \\ k - 1 \end{bmatrix}_d \right) \end{aligned}$$

which gives the second identity to be proved (with n replaced by $n + 2k - 1$, k replaced by $2k - 1$, and $r \leftrightarrow s$, $c \leftrightarrow d$ interchanged). We use above that $c_s d_s \neq 0$ for $1 \le s < m$.

Case (2). One uses

$$c^T_{(rs)_{n+2k-1},(rs)_n} \alpha_{n+1} \alpha_{n+2} \cdots \alpha_{n+k-1} \equiv \pm \alpha_0 \alpha_{-1} \cdots \alpha_{-(k-2)} \pmod{\alpha_{n+k}}$$

and $\alpha_s \equiv -F_{s,n+k} \alpha_{n+k+1} \pmod{\alpha_{n+k}}$ for $k \ge 1$ with $n + 2k - 1 < m + 1$ to prove the first identity in the Proposition.

Case (3). One uses

$$c^\emptyset_{(sr)_{n-2k+1},(rs)_n} \alpha_1 \alpha_2 \cdots \alpha_{k-1} \equiv \pm \alpha_n \alpha_{n-1} \cdots \alpha_{n-(k-1)+1} \pmod{\alpha_k}$$

and $\alpha_s \equiv F_{k,s} \alpha_{k+1} \pmod{\alpha_k}$ for $k \ge 1$ with $n - 2k + 1 \ge 0$ to prove the third identity in the Proposition.

Case (4). One uses

$$c^\emptyset_{(rs)_{n-2k+1},(rs)_n} \alpha_n \alpha_{n-1} \cdots \alpha_{n-(k-1)+1} \equiv \pm \alpha_1 \alpha_2 \cdots \alpha_{k-1} \pmod{\alpha_{n-k+1}}$$

and $\alpha_s \equiv F_{s,n-k+1} \alpha_{n-k} \pmod{\alpha_{n-k+1}}$ for $k \ge 1$ with $n - 2k + 1 \ge 0$ to prove the fourth identity in the Proposition (with $r \leftrightarrow s$ and $c \leftrightarrow d$).

∎

2.26. **A-singular loci in rank two.** Let A be a subring of R over which Φ is defined, and recall the definition 1.22 of the A-singular locus of $w \in W$ (in the order $\leq^I$ with $I = \emptyset$ or $I = T$).

Proposition. *Consider a closed subinterval Ω of W in the order leq^I. Then the folowing conditions are equivalent:*

(i) *For each $x \leq^I y$ in Ω, $[x, y]$ is A-smooth*
(ii) *For each $x \leq^i y$ in Ω with $l^I(x, y) = 2$, $[x, y]$ is A-smooth.*

Proof. That (i) implies (ii) is trivial; the reverse implication follows readily from the definitions and Proposition 2.25, on noting that if a binomial coefficient $c^I_{y,x}$ with $y \leq^I x$ in Ω is not a unit in A, then one of the factors (root coefficients) in its "numerator" (as in 2.12) must be a non-unit in A, and that the root coefficients which so appear are precisely the $c^I_{y,x}$ with $l^I(y, x) = 2$. □

2.27. We assume Φ is defined over the valuation ring A of a number field L with a non-trivial discrete valuation ν. Let $\mathfrak{m}$ be the maximal ideal of A and p (a prime in $\mathbb{Z}$) be the characteristic of the residue ring $A/\mathfrak{m}$. Let l be the minimal positive integer such that $\nu(c_l) > 0$ (i.e. $c_l \in \mathfrak{m}$); such an integer exists by Lemma 2.9. Consider the l-modified p-adic expansion $n = \sum_{i \geq 0} n_i P_i$ for an integer $n \geq 0$, as in Remark 2.15.

Corollary. *The A-singular locus of $w := (rs)_n$ in $(W, \leq^I)$ ($I = \emptyset$ or $I = T$) is determined as follows.*

(a) *Suppose first that $I = T$. If $n = 0$, then w is A-smooth. If $n > 0$, choose i minimal with $n_i \neq 0$. If $n_j \neq 0$ for all $j > i$, then w is A-smooth in $\leq^T$. Otherwise, the minimum element of the A-singular locus of w in $\leq^T$ is $(rs)_{n-2v}$ where $v = n_i P_i$.*
(b) *Now suppose that $I = \emptyset$ and rs has infinite order. If $n = 0$, w is A-smooth in $\leq^\emptyset$. If $n > 0$, choose i-minimal with $n_i \neq 0$. Then the minimum element of the A-singular locus of w in $\leq^\emptyset$ is $(rs)_{n+2v}$ where $v = (p_i - n_i)P_i$.*

Proof. This follows readily from Propositions 2.25 and 2.15 (the latter applied to the composite homomorphism $\mathbb{Z}[x, y] \xrightarrow{\phi} A \to A/\mathfrak{m}$). □

Remarks. If rs has finite order m in the situation of (b), then $w \neq 1$ is A-smooth if $n + 2v > m$, and otherwise the A-singular locus is as given in (b). However, this result is equivalent to that in (a) for finite W, since then Ω^T_W and $\Omega^\emptyset_W$ are isomorphic as edge labeled directed graphs.

3. Applications and Examples

3.1. **Smoothness of Schubert varieties.** Suppose that G is a Kac-Moody group over $\mathbb{C}$ associated as in [30] to a not necessarily symmetrizable generalized Cartan matrix (refer to loc cit here and below for precise definitions). Alternatively, we could take G as a connected, simply connected, semisimple algebraic group over the complex numbers (see e.g. [2]).

Fix a standard maximal torus and Borel subgroup $T \subseteq B \subseteq G$, and define the Weyl group $W := N_G(T)/T$ of G, where $N_G(T)$ is the normalizer of T in G. For $w \in W$, choose a representative $\dot{w}$ of W in $N(T)$. We let $\mathfrak{h} = \text{Lie}(T)$ denote the Cartan subalgebra corresponding to T of the (Kac-Moody or semisimple complex)

Lie algebra $\mathfrak{g}$ corresponding to G. Let $\Pi \subseteq \Phi_+ \subseteq \Phi \subseteq \mathfrak{h}^*$, $\Pi^\vee \subseteq \Phi_+^\vee \subseteq \Phi^\vee \subseteq \mathfrak{h}$ denote the simple roots, positive (real) roots etc corresponding to our choice of B. Taking $V = \mathbb{R}\Pi$, $V' = \mathbb{R}\Pi^\vee$ and $\iota\colon \Pi \to \Pi^\vee$ as the natural bijection, we obtain a based root datum $E = (\langle\ ,\ \rangle\colon V \times V' \to \mathbb{R}, \iota\colon \Pi \to \Pi^\vee)$ in the sense of Appendix A, with corresponding Coxeter group naturally identified with W. All results of Sections 1–2 are applicable in this situation.

Associated to the above data, we have the flag variety G/B (which is possibly infinite-dimensional in the Kac-Moody setting) and (finite-dimensional, complex projective) Schubert varieties $X_w := \overline{B\dot{w}B/B}$ in G/B for $w \in W$, namely the closures of the B-orbits on G/B in their natural reduced scheme structure. We have $X_v \subseteq X_w$ iff $v \leq^\emptyset w$ iff $w \leq^T v$ where $\leq^\emptyset$ (resp., $\leq^T$) denotes ordinary (resp., reverse) Chevalley-Bruhat order on W. We identify W with the set of T-fixed points of G/B by the map $w \mapsto \dot{w}B$. For $I = \emptyset$ or T, we write $[v,w]^I := \{\, z \in W \mid v \leq^I z \leq^I w \,\}$.

3.2. For definitions and some discussion of rational smoothness and smoothness of points of Schubert varieties, see for instance [1], [27], [5] and [6]. We record in the notation of this paper the main result of Kumar [31, 5.5], which characterizes the smooth points and rationally smooth points of Schubert varieties X_w.

Theorem. *Define $c_{y,x} = c^T_{y,x} \in S(V)$ for $y \leq^T x$ as in* 1.6*. For $y \leq^T x$ in W,*

(a) *x is a rationally smooth point of X_y iff $c_{y,z} \in \mathbb{R}$ for all z with $y \leq^T z \leq^T x$*
(b) *x is a smooth point of X_y iff $c_{y,x} = 1$*

Remarks. Since x is smooth in X_y iff z is smooth in X_y for all $y \leq^T z \leq^T x$, the condition $c_{y,x} = 1$ in (b) is equivalent to $c_{y,z} = 1$ for all z with $y \leq^T z \leq^T x$. Since Φ is defined over $\mathbb{Z}$, $c_{y,x} \in \mathbb{Z}[\Pi] \subseteq S(V)$ for any $y \leq^T x$ by Lemma 1.9.

3.3. Recall from 1.22 that for $I = \emptyset$ or $I = T$, we say that $[y,x]^I$ is A-smooth if $y \leq^I x$ and $c^I_{y,z}$ is a unit of A for all $z \in [y,x]^I$. Kumar's criteria imply that rational smoothness (resp., smoothness) of x in X_y is equivalent to $\mathbb{R}$-smoothness (resp., $\mathbb{Z}$-smoothness) of $[y,x]^T$ in this sense. We shall say that x is an A-smooth point of X_y if $[y,x]^T$ is A-smooth. The next result follows immediately from Theorem 3.2 using Theorems 1.11 and 1.14.

Theorem. *Let A be any subring of $\mathbb{R}$.*

(a) *For $v \leq^\emptyset w$ in W, v is a rationally smooth point of X_w iff for each coset Dz of each maximal dihedral reflection subgroup D of W such that $z \in [v,w]^\emptyset$ and $\sharp(Dz \cap [v,w]^\emptyset) \geq 3$, there is a unique $x \in Dz \cap [v,w]^\emptyset$ such that for each reflection t of D, either $tx <^\emptyset x$ or $tx \not\leq^\emptyset w$.*
(b) *Fix $v \in W$ with v rationally smooth in X_w. For each D, z and x as in* (a)*, let y be the minimum element of Dz in the order $\leq^\emptyset$ on W, and let H be a Kac-Moody group (as in* [30]*) with the same root system as D and standard Borel subgroup C corresponding to the roots for D which are positive for B. Then v is an A-smooth point of X_w iff for each Dz (and the corresponding x, y, H, C), zy^{-1} is an A-smooth point in $\overline{C\dot{x}\dot{y}^{-1}C/C}$.*

3.4. To apply the above theorem to investigate smoothness of a point $v \in X_w$, it is only necessary to consider the finitely many maximal dihedral subgroups D of W for which $D \cap [w,v]^T$ has at least three points; these are effectively computable (e.g using results of [15] adapted to the more general class of root systems considered here (cf Appendix A). The necessary information on singular loci of Schubert varieties for rank two Kac-Moody groups is recorded in 3.7–3.13.

For G of finite or affine type above, it is easy to describe all the maximal dihedral subgroups of W completely. In the finite case, they are just the rank two parabolic subgroups (for the finite classical types A–D, one can even find all reflection subgroups of W listed explicitly in [14]). In the affine case, the maximal dihedral subgroups are the rank two parabolic subgroups together with certain infinite dihedral subgroups; in the irreducible affine case, a typical infinite maximal dihedral subgroup is generated by all reflections in reflecting hyperplanes parallel to a fixed reflecting hyperplane, in the standard representation of W as affine reflection group.

For G of classical (finite) type, the cosets of the maximal dihedral subgroups of W have simple explicit descriptions using standard representations (by (signed) permutations, for instance) of elements of W. Using the known descriptions of singular loci for rank two finite groups, and standard descriptions of Chevalley-Bruhat order in terms of signed permutations etc (collected in [1], for example), it would be routine to write down explicit combinatorial characterizations of singular loci in these cases (see also 3.17).

3.5. We give the following generalization of Peterson's A, D, E-theorem [6].

Corollary. *Let G be a semisimple, simply connected complex algebraic group and let $x \in W$ be a rationally smooth T-fixed point of X_y.*

(a) *In general, $c_{y,x} = 2^n 3^m$ for some $n, m \in \mathbb{N}$.*
(b) *If G has no factors of type G_2, then $c_{y,x} = 2^n$ for some $n \in \mathbb{N}$*
(c) *If G is simply laced, then $c_{y,x} = 1$.*

Proof. This follows directly by induction from Lemma 1.13 and the well-known values of the $c_{y,x}$ for the rank two semisimple groups (see [31], [1] or 3.7–3.10). □

Remarks. According to [31, Remark 5.3], if there are no components of G of type G_2, then $c_{y,x}$ is the multiplicity of x in X_y. We do not know if this holds for G_2 or in the general Kac-Moody setting.

3.6. **Examples.** Recall the terminology 1.22. In subsections 3.7-3.13, we determine the singular loci in that sense of points $w \in W$ for both orders $\leq^T$, $\leq^\emptyset$ and the root systems associated to the generalized Cartan matrices $\left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ of rank two (for the finite types, we treat only $\leq^T$ since the results for $\leq^\emptyset$ are equivalent). We repeat that v is a smooth point of the Schubert variety X_w iff v is a smooth point of w in reverse Chevalley-Bruhat order $\leq^T$. Additionally, we shall say p-smooth, p-singular etc instead of $\mathbb{Z}_{(p\mathbb{Z})}$-smooth, $\mathbb{Z}_{(p\mathbb{Z})}$-singular etc, where $\mathbb{Z}_{(p\mathbb{Z})}$ denotes the localization of $\mathbb{Z}$ at its prime ideal $p\mathbb{Z}$ for a prime integer p (as in the Introduction).

For each of the rank two root systems considered, we denote the simple roots as A, B and the corresponding simple reflections as r and s respectively. We write m for the order of the product rs, and for a positive prime integer p, set $l_p := \min\{\, k \in \mathbb{N}_{>0} \mid p|c_q \,\}$, $l'_p := \min\{\, q \in \mathbb{N}_{>0} \mid p|d_q \,\}$. For the finite types ($ab < 4$), we shall explicitly list the p-singular loci and singular loci. For the infinite types $ab \geq 4$, we are unable to improve on Proposition 2.27, except for explicitly giving the values of l_p and l'_p in the affine cases $ab = 4$. In the finite (resp., affine) case, we also list (resp., express in terms of classical binomial coefficients) all elements $\phi\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right)_z$ with $z = c$ or $z = d$ and $0 \leq k \leq n < m$ (resp., $k \geq 0$)

where $\phi\colon \mathbb{Z}[x,y] \to \mathbb{Z}$ is the ring homomorphism determined by $\phi(x) = a$, $\phi(y) = b$. Finally, for the infinite types ($ab \geq 4$) we describe the singular loci in both orders $\leq_\emptyset$ and $\leq_T$. Recall that all elements of (rank two) W are $\mathbb{R}$-smooth in either order $\leq_\emptyset$ or $\leq_T$. Most of the facts follow directly from the general results, so will be stated without proof; many are well known, especially in the finite cases.

For singular w, we shall indicate the singular locus of w in the order $\leq^I$, when it is non-empty, simply by listing the minimum element (in $\leq^I$) of the singular locus.

3.7. $A_1 \times A_1$**.** We have $a = b = 0$, $\Phi_+ = \{A, B\}$, $m = 2$, $(c_n)_{n=0}^2 = (d_n)_{n=0}^2 = (0,1,0)$, $\phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]\big)_c = \phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]\big)_d = 1$ for $0 \leq k \leq n < 2$. Further, $l_p = l'_p = 2$ for any prime p. All $w \in W$ are p-smooth and smooth in $\leq_T$.

3.8. A_2**.** We have $a = b = 1$, $\Phi_+ = \{A, A+B, B\}$, $m = 3$, $(c_n)_{n=0}^3 = (d_n)_{n=0}^3 = (0,1,1,0)$, $\phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]\big)_c = \phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]\big)_d = 1$ for $0 \leq k \leq n < 3$. Further, $l_p = l'_p = 3$ for any prime p. All $w \in W$ are p-smooth and smooth in $\leq_T$.

3.9. B_2**.** We take $a = 1$, $b = 2$, $\Phi_+ = \{A, A+B, A+2B, B\}$, $m = 3$, $(c_n)_{n=0}^4 = (0,1,1,1,0)$, $(d_n)_{n=0}^4 = (0,1,2,1,0)$, $\phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]_c\big) = \phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]_d\big) = 1$ for all $0 \leq k \leq n < 4$ except $\phi\big(\big[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big]_d\big) = 2$. For $p \neq 2$, $l_p = l'_p = 4$ and all $w \in W$ are p-smooth in $\leq_T$. For $p = 2$, $l_2 = 4$ and $l'_2 = 2$ and all w are 2-smooth and smooth in $\leq_T$ except srs with 2-singular locus (and singular locus) s.

3.10. G_2**.** We take $a = 1$, $b = 3$, $\Phi_+ = \{A, A+B, 2A+3B, A+2B, A+3B, B\}$, $m = 6$, $(c_n)_{n=0}^6 = (0,1,1,2,1,1,0)$, $(d_n)_{n=0}^6 = (0,1,3,2,3,1,0)$. For $p > 3$, we have $l_p = l'_p = 6$ and all w are p-smooth in $\leq_T$. For $p = 3$, we have $l_3 = 6$ and $l'_3 = 2$, while for $p = 2$ we have $l_2 = l'_2 = 3$. We give the other information in Tables 1–2 below.

$\phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]_c\big)$	$n=0$	1	2	3	4	5
$k=0$	1	1	1	1	1	1
1		1	1	2	1	1
2			1	2	2	1
3				1	1	1
4					1	1
5						1

$\phi\big(\big[\begin{smallmatrix} n \\ k \end{smallmatrix}\big]_d\big)$	$n=0$	1	2	3	4	5
$k=0$	1	1	1	1	1	1
1		1	3	2	3	1
2			1	2	2	1
3				1	3	1
4					1	1
5						1

TABLE 1. Specializations of certain binomial coefficients in type G_2; missing entries indicate zeros.

Element	2-singular locus	3-singular locus	singular locus
srs	$\emptyset$	s	s
$rsrs$	rs	$\emptyset$	rs
$srsr$	sr	$\emptyset$	sr
$rsrsr$	r	$\emptyset$	r
$srsrs$	s	srs	srs

TABLE 2. The singular w in $\leq^T$ in type G_2, and their singular loci, 2-singular loci and 3-singular loci

3.11. $A_1^{(1)}$**.** Here, $a = b = 2$, $c_n = d_n = n$ for $n \in \mathbb{Z}$, and $\phi\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_z\right) = \binom{n}{k}$ for $z = c$ or $z = d$ and all integers n and k. For a prime p, we have $l_p = l'_p = p$.

3.12. $A_1^{(2)}$**.** Here, we take $a = 1$, $b = 4$. We have

$$c_n = \begin{cases} \frac{n}{2} & \text{for even } n \\ n & \text{for odd } n \end{cases}, \qquad d_n = \begin{cases} 2n & \text{for even } n \\ n & \text{for odd } n \end{cases}$$

$$\phi\left(\begin{bmatrix} n \\ k \end{bmatrix}_c\right) = \begin{cases} \frac{1}{2}\binom{n}{k} & \text{for odd } k(n-k) \\ \binom{n}{k} & \text{for even } k(n-k) \end{cases}$$

$$\phi\left(\begin{bmatrix} n \\ k \end{bmatrix}_d\right) = \begin{cases} 2\binom{n}{k} & \text{for odd } k(n-k) \\ \binom{n}{k} & \text{for even } k(n-k). \end{cases}$$

For a prime p, we have $l_p = l'_p = p$ if $p > 2$, and $l_2 = 4$, $l'_2 = 2$. Thus, the pattern of p-smooth loci is the same as for $A_1^{(1)}$ provided $p \neq 2$.

3.13. We shall call $\mathbb{Z}$-smoothness in $\leq^I$ simply smoothness in $\leq^I$, etc. Finally, we describe the singular loci in the infinite cases.

Proposition. *Suppose that $ab \geq 4$ with $b \geq a$.*

(a) *Suppose $a \geq 2$. Then the smooth w in $\leq_T$ are 1, r, s, rs and sr; the singular locus of $(rs)_n$ (resp., $(sr)_n$) in the order $\leq_T$ is $(rs)_{n-2}$ (resp., $(sr)_{n-2}$ for $n \geq 3$.*

(b) *If $a \geq 2$, the only smooth w in the order $\leq_\emptyset$ is 1, and the singular locus of $(rs)_n$ (resp., $(sr)_n$) in the order $\leq_\emptyset$ is $(rs)_{n+2}$ (resp., $(sr)_{n+2}$) for $n \geq 1$.*

(c) *If $a = 1$, then the smooth w in $\leq_T$ are 1, r, s, rs, sr and rsr; the singular locus of $(rs)_n$ (resp., $(sr)_n$) in the order $\leq_T$ is $(rs)_{n-2}$ (resp., $(sr)_{n-2}$ for $n \geq 4$ (resp., $n \geq 3$).*

(d) *If $a = 1$, the only smooth w in the order $\leq_\emptyset$ is 1, and the singular locus of $(rs)_n$ (resp., $(sr)_n$) in the order $\leq_\emptyset$ is $(rs)_{n+2}$ (resp., $(sr)_{n+2}$) for $n \geq 1$ (resp., $n \geq 2$) while that of s is $(sr)_5$.*

Proof. If $(rs)_n$ is singular in the order $\leq_T$ (resp., $\leq_\emptyset$) its singular locus is necessarily of the form $(rs)_{n-2v}$ (resp., $(rs)_{n+2v}$) for some $v \geq 1$ (by Lemma 2.27, for example). Using (2.2.6) and Proposition 2.25, one readily checks that the singular loci are contained in the indicated sets in each case. In the affine cases, it is clear from 3.11–3.12 that the singular loci are as given. In the general case, we observe from (2.11.5) that for positive $a, b \in \mathbb{R}$ with $ab \geq 4$, $C_n(a,b)$ and $D_n(a,b)$ are both non-decreasing functions of a and both non-decreasing as functions of b; it follows from (2.12.3) that $\phi\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_z\right)$ is also non-decreasing as a function of either a or b in this range, for $z = c$ or $z = d$ and for $0 \leq k \leq n$. This implies that the singular loci in cases (a) and (c) (resp., (b) and (d)) have the same pattern as for the affine case $A_1^{(1)}$ (resp., $A_1^{(2)}$). $\square$

3.14. **Connection to representation theory.** We indicate without proof or even complete statements some of the connections of the subject matter of this paper with representation categories [21] and generalizations (see also [12, 13]).

Adopt the notation of 1.19. Fix an interval $X \in \mathcal{P}^I$. Give X the order topology, with a basis for open sets provided by the sets $\{w \mid w \leq^I x\}$ for $x \in X$. Let $A[\Pi]$ be the $\mathbb{Z}$-graded subring of $S(V)$ generated by $A \cup \Pi$ (it is a polynomial ring over

A on generators Π), graded so $A[\Pi]_0 = A$ and $A[\Pi]_2 = A\Pi := \sum_{\alpha\in\Pi} A\alpha \subseteq V$. Let Z be the naturally-graded $A[\Pi]$-algebra of functions $f\colon X \to A[\Pi]$ satisfying $f(x) \equiv f(y) \pmod{\alpha A[\Pi]}$ for all $x \xrightarrow{\alpha} y$ in $\Omega_W^I(x)$, under pointwise operations. For $x \in X$, let $\epsilon_x\colon Z \to A[\Pi]$ denote the evaluation homomorphism $f \mapsto f(x)$. One can verify that the data $(Z, A[\Pi], A[\Pi], X, \epsilon := \{\epsilon_x\}_{x\in X}, l^I)$ satisfies the conditions in [21, 1.12] and one may define the exact category $\mathcal{C}$-gr as there. From [13], it follows that there is a projective object P of $\mathcal{C}$-gr such that any projective object of $\mathcal{C}$-gr is a direct summand of a finite direct sum of degree shifts of P (if A is local, P could by loc cit be uniquely specified by conditions as in [21, Proposition 1.12], where however the statement was limited to the case that A is a field). We define the $\mathbb{Z}$-graded $A[\Pi]$-algebra $\mathcal{B}_A := \mathrm{End}(P)^{\mathrm{op}}$, well-defined up to Morita equivalence.

Remarks. We consider this construction in greater generality elsewhere. As one of several very interesting variants related to Coxeter groups, we mention the closely related K-theory analogue which is defined if $A = \mathbb{Z}$; replace $\mathbb{Z}[\Pi]$ by the integral group ring $\mathbb{Z}[L]$ of the root lattice $L = \mathbb{Z}\Pi$, use $f(x) \equiv f(y) \pmod{(e^\alpha - 1)\mathbb{Z}[L]}$ in the definition of Z, and use filtrations in place of gradations. Lemma 1.9 and its K-theory analogue become more natural when interpreted in the exact category e.g. 1.9(c) generalizes to arbitrary objects of $\mathcal{C}$-gr, where it follows from an Ext^1-vanishing property.

3.15. For a maximal ideal $\mathfrak{m}$ of A, let $B := A_\mathfrak{m}$ be the localization of A at $\mathfrak{m}$ and $k = A/\mathfrak{m}$ be the residue field of the (local) ring B, regarded as $B[\Pi]$-module annihilated by Π; then (see [13]) $\mathcal{B}_B \otimes_{B[\Pi]} k$ is a finite-dimensional $\mathbb{Z}$-graded k-algebra with a highest weight representation theory (it is quasi-hereditary (see [7]) with weight poset X as ungraded k-algebra, for example). It has (suitably graded) simple modules L_x^B and universal highest weight modules (Verma modules) $\overline{\Delta}_x^B$ naturally parameterized by X. We form Poincaré polynomials

$$m_{y,x}^{B,W,I} = m_{y,x}^{B,W,I}(v) := \sum_{n\in\mathbb{Z}} [\overline{\Delta}_x^B : L_y^B\langle n\rangle] v^n \in \mathbb{Z}[v, v^{-1}],$$

from the graded composition factor multiplicities of $L_y^B\langle n\rangle$ in $\overline{\Delta}_x^B$, where $\langle n\rangle$ denotes grading shift. The theorem below, in which (a)–(c) are essentially trivial in the appropriate context, collects some properties of these multiplicities which we shall prove elsewhere; in particular, (e) provides a representation-theoretic interpretation of B-smoothness for local rings B.

Theorem. *Let $B \subseteq \mathbb{R}$ be any Noetherian local ring over which Φ is defined.*

(a) $m_{y,z}^{B,W,I} = 0$ *unless* $y \leq^I z$; $m_{y,y}^{B,W,I} = 1$.

(b) $m_{y,z}^{B,W,I} \geq m_{y,z}^{B',W,I}$ *(coefficientwise) if B' is a Noetherian local ring with $B \subseteq B' \subseteq \mathbb{R}$, and for fixed y, the set of z for which strict inequality holds is closed in X.*

(c) *If A is the ring of algebraic integers of a number field, than* $m_{y,x}^{A_\mathfrak{m},W,I} = m_{y,x}^{\mathbb{R},W,I}$ *for all but finitely many maximal ideals $\mathfrak{m}$ of A.*

(d) $m_{y,x}^{B,W,I} \geq v^{l^I(z,x)} m_{y,z}^{B,W,I}$ *(coefficientwise) for $z \in [y,x]^I$.*

(e) *The ungraded composition factor multiplicity $m_{y,x}^{B,W,I}(1) \in \mathbb{N}$ of L_y^B in $\overline{\Delta}_x^B$ is equal to 1 iff $[y,x]^I$ is B-smooth in the sense of this paper.*

3.16. For the remarks here, we need to consider root systems for which the simple roots are not necessarily linearly independent, or finite in number (see the Appendix A). Let us say that a subgroup W' of W is pseudoparabolic (for the given reflection representation of W with root system Φ) if it is generated by $\{ s_\alpha \mid \alpha \in U \cap \Phi \}$ for some linear subspace U of $\mathbb{R}\Pi$ (for finite W, the pseudoparabolic subgroups are just the parabolic subgroups). There are then inequalities such as

$$m_{y,x}^{B,W,I}(1) \geq \sum_z m_{y,z}^{B,W,I}(1) m_{zp^{-1},xp^{-1}}^{B,W',p\cdot I\cap W'}(1) \tag{3.16.1}$$

where z ranges over the vertices z of $\Omega^I([y,x] \cap W'x)$ for which there is some directed path from z to x in $\Omega^I([y,x]\cap W'y)$ but no edge $t \xrightarrow{\alpha} z$ in $\Omega^I([y,x]\cap W'y)$, and $p \in W'x = W'z$ is fixed but arbitrary. These arise as follows. Let B' be the localization $B[\Pi]_\mathfrak{p}$ of $B[\Pi]$ at the muliplicative set of all homogeneous elements of $B[\Pi]$ not lying in its homogeneous prime ideal $\mathfrak{p}$ generated by the maximal ideal of B and $U \cap \mathbb{Z}\Pi$; B' is chosen so the weight posets of the blocks of $\mathcal{B}' := \mathcal{B}_B \otimes_{B[\Pi]} B'$ are the non-empty connected components of the graphs $\Omega(X \cap W'z)$ as $W'z$ ranges over cosets of W'. The above inequality reflects the decomposition $P_y^{B,W,I} \otimes_{B[\Pi]} B' = \oplus n_{y,z} P_z^{B,W',I} \otimes_{B[\Pi]} B'\langle l^I(y,z)\rangle$ of the localization of a projective indecomposable $P_y^{B,W,I}$ of $\mathcal{B} = \mathcal{B}^W$ as a direct sum of localizations of projective indecomposables $P_z^{B,W',I}$ for a similar algebra $\mathcal{B}^{W'z}$ associated similalrly to a coset of W'. Considering filtration multiplicities of standard objects parametrized by z in both sides shows $n_{y,z} = m_{y,z}^{B,W,I}(1)$ for z as in the sum in (3.16.1), and then (3.16.1) follows by looking at the filtration multiplicities of standard objects coresponding to x. One has $B' \cong B'\langle 2\rangle$, so one loses information on graded multiplicities.

Let us indicate how (3.16.1) specializes if W is a finite Weyl group, $A = \mathbb{R}$, $\mathfrak{m} = 0$ and $I = T$. The proven Kazhdan-Lusztig conjecture for semisimple complex Lie algebras then implies that $m_{y,x}^{\mathbb{R},W,T} = v^{l^T(y,x)} P_{x,y}(v^{-2})$ where $P_{x,y}$ denotes a Kazhdan-Lusztig polynomial [27] (see [21]; this equality conjecturally holds for arbitrary W). Then (3.16.1) becomes

$$P_{x,y}(1) \geq \sum_z P_{xq^{-1},zq^{-1}}^{W'}(1) P_{z,y}(1) \tag{3.16.2}$$

where z runs over the same elements as before, where q is the maximum element of $W'x$ in $\leq^T$ and where the $P^{W'}$ are Kazhdan-Lusztig polynomials computed in W', with respect to its canonical Coxeter generators. These last inequalities (for finite Weyl groups) have been obtained independently (by a technique of "hyperbolic localization in intersection cohomology") in [4].

3.17. The inequalities (3.16.1) are far from equalities and are probably not of any particular significance except for the following special case. In (3.16.1), each term on the right is one or greater, so if $m_{y,x'}^{B,W,I}(1) = 1$ for some $x' \geq x$, then by Theorem 3.15(d) there is only one term in the sum on the right of (3.16.1) (and that term is necessarily equal to one). Applying this remark with $B = \mathbb{R}$ gives part (a) of the following:

Theorem. (a) *If $[y,x]^I \in \mathcal{P}^I$ is $\mathbb{R}$-smooth, then for each pseudoparabolic subgroup W' of W and each $z \in [y,x]^I$, there is a unique vertex z' of $\Gamma := \Omega^I([y,x] \cap W'z)$ such that there is a directed path from z' to z in Γ but no edge $z'' \xrightarrow{\alpha} z'$ in Γ.*

(b) *Let A be a subring of $\mathbb{R}$ and M' be any family of pseudoparabolic subgroups of W, such that any dihedral reflection subgroup of W is contained in some element of M'. Then $[y,x]^I \in \mathcal{P}^I$ is A-smooth iff for each W' in M' and $z \in [y,x]^I$ as in* (a)*, there is a unique z' as in* (a) *and moreover $[z'p^{-1}, zp^{-1}]_{W'}^{p\cdot I\cap W'}$ is A-smooth in $\leq_{W'}^{p\cdot I\cap W'}$ for some (or equivalently, any) $p \in W'z = W'z'$.*

We are unable to give a direct combinatorial proof of the statement (a) along the lines of the arguments in Sections 1–2 this paper. However, given (a), part (b) follows readily by applying Theorems 1.11 and 1.14 to W and the various W'.

For finite Weyl groups, (a) also follows from the above-mentioned work [4] and then (b) with $A = \mathbb{Z}$ provides criteria for smoothness of points v in Schubert varieties X_w for semisimple complex algebraic groups, in terms of smoothness of points in Schubert varieties for lower rank groups. When formulated concretely in terms of (signed) permutations for classical types A, B, C, D, the resulting smoothness criteria for the case M' consists of a suitable family of rank three parabolic subgroups resembles and is related to (but does not obviously imply) various known and conjectural characterizations of singular loci in terms of "patterns" (cf [1, Chapter 8], [4]).

For crystallographic W , (b) with $A = \mathbb{Z}$ can be regarded as giving similar 'pattern" descriptions of smooth and rationally smooth loci of Schubert varieties for Kac-Moody groups. For more general W and B, (b) may be reagrded as giving similar "pattern" descriptions of multiplicity one loci for the representation categories mentioned above.

Remarks. We have not attempted to compare the efficiency of the smoothness criteria here with other general and special tests for smoothness. However, we remark that several general tests require (at least implicitly) a preliminary test for rational smoothness, and the efficiency of this step might be considerably improved if some statement akin to 1.15 could be proved.

3.18. The determination of the multiplicities $m_{y,z}^{B,W,I}$ even in special cases such as for finite or affine Weyl groups is a very interesting problem. The results proved in this paper together with Theorem 3.15 provide a description of the closed set $M_{y,B} := \{z \geq y \mid m_{y,z}^{B,W,I}(1) > 1\}$ of the closure $\overline{y}$ of y. Conjecturally, $m_{y,z}^{\mathbb{R},W,I}$ is expressible in terms of the appropriate Kazhdan-Lusztig polynomials which are defined purely combinatorially as in [27], [28], [17] from W and the order $\leq^I$. It would therefore be of considerable interest to have a characterization of the "B-indifference locus" $O_{y,B} := \{z \in \overline{y} \mid m_{y,z}^{B,W,I} = m_{y,z}^{\mathbb{R},W,I}\}$, which is open in $\overline{y}$. Theorem 3.15 together with the results proved in this paper imply that $O_{y,B} \setminus M_{y,\mathbb{R}}$ can be determined by reduction to the case of dihedral groups; it is natural (but possibly too optimistic) to ask if there is a similar description of $O_{y,B}$ itself.

If W is the Weyl group of a Kac-Moody group G, it would also be very interesting to have an interpretation of $m_{y,x}^{B,W,T}$, in terms of geometric data attached to the point x in the Schubert variety X_y, similar to the interpretation [28] of Kazhdan-Lusztig polynomials as Poincaré series of the local intersection cohomology (for $B = \mathbb{R}$, this would follow immediately from the above-mentioned conjecture that $m_{y,x}^{\mathbb{R},W,T} = v^{l^T(y,x)} P_{x,y}(v^{-2})$). In general, $m_{y,x}^{B,W,T}$ cannot arise directly as a Poincaré series of local intersection cohomology since, although satisfying the equations corresponding to Verdier duality (fixedness under the Kazhdan-Lusztig involution) it does not satisfy the necessary degree-vanishing conditions (the analogue of the degree bounds for Kazhdan-Lusztig polynomials). Possibly it has in some cases an interpretation in a framework similar to that considered in [34, Section 3] or perhaps in connection with some currently unknown K-theory analogue of categories of perverse sheaves (cf Remark 3.14 and [21, 8.3]).

Remarks. Apart from combinatorial and geometric applications and their intrinsic representation-theoretic interest, another main reason we have considered these representation categories is the possibility that, in the case of affine Weyl groups, they are related to categories of representations of semisimple algebraic groups in defining characteristic $p > 0$ (and quantum groups at roots of unity, in the K-theory case). Relations between $m_{x,y}^{\mathbb{Z}_{(p\mathbb{Z})},W,T}(1)$ (for W a finite Weyl group and p larger than the Coxeter number) and special multiplicities in semisimple algebraic groups can be deduced using results in [34]. One might hope that the categories considered here for affine Weyl groups are related to regular block categories for semisimple algebraic groups in an analogous manner to that in which regular blocks of $\mathcal{O}$ are related to blocks of parabolic $\mathcal{O}$ for semisimple complex Lie algebras, at least for sufficiently large p; the Kazhdan-Lusztig conjecture and Koszulity conjecture in characteristic 0 in [21] together with the Lusztig conjecture would imply the existence of such a relation at the level of multiplicities (for $p >> 0$).

Appendix A. Root systems and reflection representations

We describe our technical assumptions on the reflection representations and corresponding root systems which we consider in this paper. The results are variants of standard facts and can can be proved in a similar way (see [9],[15],[14]).

A.1. We consider two $\mathbb{R}$-vector spaces V, V' with a given fixed $\mathbb{R}$-bilinear pairing $\langle\ ,\ \rangle\colon V \times V' \to \mathbb{R}$. For any $\alpha \in V$, $\alpha' \in V$ with $\langle \alpha, \alpha' \rangle = 2$, we let $s_{\alpha,\alpha'} \in \mathrm{GL}(V)$ be the linear map (pseudoreflection) given by $v \mapsto v - \langle v, \alpha' \rangle \alpha$, and define $s_{\alpha',\alpha} \in \mathrm{GL}(V')$ similarly. Assume given subsets $\Pi \subseteq V$, $\Pi^\vee \subseteq V'$ such that Π and $\Pi^\vee$ are $\mathbb{R}$-linearly independent, or more generally, such that each finite subset of Π (resp., Π') forms a set of representatives for the extreme rays of a pointed polyhedral cone in V (resp., V') i.e. if $\sum_{\alpha\in\Pi} c_\alpha \alpha = 0$ with $c_\alpha \in \mathbb{R}$ and at most one c_α negative, then all $c_\alpha = 0$, and similarly for $\Pi^\vee$. We also assume there is a given bijection $\iota\colon \Pi \to \Pi^\vee$ denoted $\alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for $\alpha \in \Pi$. Let $R := \{\, s_{\alpha,\alpha^\vee} \mid \alpha \in \Pi \,\}$, W denote the subgroup of $\mathrm{GL}(V)$ generated by R, $\Phi := \cup_{w\in W} w(\Pi)$, and $\Phi_+ := \Phi \cap \sum_{\alpha\in\Pi} \mathbb{R}_{\geq 0}\alpha$ Define R', W', $\Phi^\vee$, $\Phi^\vee_+$ similarly using $\Pi^\vee \subseteq V'$ instead of $\Pi \subseteq V$.

Define $P := \{\, 4\cos^2 \frac{\pi}{m} \mid m \in \mathbb{N}_{\geq 2} \,\} \cup [4, \infty) \subseteq \mathbb{R}_{\geq 0}$. In the above setting, the following conditions (i)–(iii) can be shown to be equivalent:

(i) $\Phi = \Phi_+ \cup (-\Phi_+)$
(ii) $\Phi^\vee = \Phi^\vee_+ \cup (-\Phi^\vee_+)$
(iii) for $\alpha \neq \beta$ in Π, we have $\langle \alpha, \beta^\vee \rangle \leq 0$ and $c_{\alpha,\beta} := \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in P$; moreover $\langle \alpha, \beta^\vee \rangle = 0$ iff $\langle \beta, \alpha^\vee \rangle = 0$

We say that $E = (\langle\ ,\ \rangle\colon V \times V' \to \mathbb{R}; \iota\colon \Pi \to \Pi^\vee)$ is a based root datum with associated Coxeter system (W, R) if (i)–(iii) hold, as we now assume. We call $\Pi \subseteq \Phi_+ \subseteq \Phi$ the simple roots, positive roots and roots respectively, and $\Pi^\vee \subseteq \Phi^\vee_+ \subseteq \Phi^\vee$ the simple coroots, positive coroots and coroots, respectively.

For $\alpha, \beta \in \Pi$, define $m_{\alpha,\beta} = 1$ if $\alpha = \beta$, $m_{\alpha,\beta} = \infty$ if $c_{\alpha,\beta} \geq 4$ and $m_{\alpha,\beta} = m$ if $c_{\alpha,\beta} = 4\cos^2 \frac{\pi}{m}$ with $m \in \mathbb{N}_{\geq 2}$. Then (W, R) and (W', R') are isomorphic Coxeter systems with Coxeter matrix $(m_{\alpha,\beta})_{\alpha,\beta\in\Pi}$, an isomorphism being given by $\theta : s_{\alpha,\alpha^\vee} \mapsto s_{\alpha^\vee,\alpha}$ for $\alpha \in \Pi$. Regarding θ as an identification, we have $\langle w\alpha, \beta \rangle = \langle \alpha, w^{-1}\beta \rangle$ (i.e. the representation of W on V and V' are "contragredient") and the bijection $\Pi \mapsto \Pi^\vee$ extends to a W-equivariant bijection $\Phi \to \Phi^\vee$, which we still denote as $\alpha \mapsto \alpha^\vee$. The reflection in a root α or corresponding coroot $\alpha^\vee$ will be denoted just s_α.

A.2. We shall say that Φ is defined over a subring A of $\mathbb{R}$ if $\langle \alpha, \beta^\vee \rangle \in A$ for all $\alpha, \beta \in \Pi$, or equivalently, for all $\alpha, \beta \in \Phi$; then $\Phi \subseteq A\Pi$, and $\Phi^\vee$ is also defined over A. We say that Φ is reduced iff $\alpha, c\alpha \in \Phi$ with $c \in \mathbb{R}$ implies $\alpha = \pm 1$. This holds iff $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle$ for all $\alpha, \beta \in \Pi$ for which $m_{\alpha,\beta}$ is an odd integer, and also iff $\Phi^\vee$ is reduced. A root system defined over $\mathbb{Z}$ is automatically reduced in the sense here.

A.3. **Reflection Subgroups.** A subgroup W' of W which is generated by the reflections it contains (i.e. by $W' \cap T$) is called a reflection subgroup of W. The following theorem can be deduced from corresponding results in [14] or [15] or proved in the same way

Proposition. (a) *Let W' be any reflection subgroup of W. There is a set of Coxeter generators R' for W' and a based root datum $E = (\langle\, , \rangle \colon V \times V' \to \mathbb{R}; \iota' \colon \Pi' \to \Pi'^\vee)$ with associated Coxeter system (W', R') such that $\Pi' \subseteq \Phi_+$ and ι' is the restriction of $\iota \colon \Phi^\vee \to \Phi'^\vee$. The set Π' is unique up to multiplication of its elements by (possibly different) positive scalars; hence it is unique if Φ is reduced. In any case, R' is uniquely determined by W'.*

(b) *A subset Π' of Φ arises from some W as in* (a) *for some W' iff the conditions of* (iii) *above hold for all $\alpha \neq \beta \in \Pi'$.*

Remarks. This would not be true in general if in our definition of a based root datum, we did not allow Π to be infinite and possibly linearly dependent. It would also not be true even for dihedral W' if we used instead of P the set $\{ 4\cos^2 \frac{\pi}{m} \mid m \in \mathbb{N}_{\geq 2} \cup \{\infty\} \}$ (with $\frac{\pi}{\infty} := 0$) in the definition (as one effectively does for the "standard" reflection representation as in [3] or [23]).

A.4. **Reduction to the case of linearly independent roots.** Throughout Sections 1–3 of this paper, we have considered only based root datums for which Φ is reduced and Π is linearly independent, unless otherwise indicated. However, the assumed linear independence of simple roots becomes a technical inconvenience when, in order to study objects associated to W, one needs to apply results for corresponding objects defined from reflection subgroups of W. We indicate here a technique, used in the proof of the version of Theorem A.3 in [14], which can in many cases be used to reduce the general case to the case of linearly independent roots.

Consider two based root datums $E = (\langle\langle\, , \rangle\rangle \colon V \times V' \to \mathbb{R}; \iota' \colon \Pi' \to \Pi'^\vee)$ and $\tilde{E} = (\langle\, , \rangle \colon \tilde{V} \times \tilde{V}' \to \mathbb{R}; \tilde{\iota} \colon \tilde{\Pi} \to \tilde{\Pi}^\vee)$ We say that $\tilde{E}$ is a covering root datum of E if there is a given pair of linear maps $\pi \colon \tilde{V} \to V$ and $\pi' \colon \tilde{V}' \to V'$ such that $\langle \pi v, \pi' v' \rangle = \langle\langle v, v' \rangle\rangle$ for all $v \in \tilde{V}$, $v' \in \tilde{V}$, and if moreover π (resp., π') induces a bijection $\tilde{\Pi} \to \Pi$ (resp., $\tilde{\Pi}' \to \Pi'$ satisfying $\pi'(\iota(\alpha) = \tilde{\iota}(\pi(\alpha))$ for all $\alpha \in \tilde{\Pi}$.

It follows that π restricts to a bijection of root systems $\tilde{\Phi} \to \Phi$ and there is an isomorphism of corresponding Coxeter systems $\theta\colon (\tilde{W}, \tilde{R}) \cong (W, R)$ determined by $s_\alpha \mapsto s_{\pi\alpha}$ for $\alpha \in \Phi$ and satisfying $\pi(wv) = \theta(w)\pi(v)$ for $w \in W$ and $v \in V$ (similarly for W acting on V'). Given any based root datum E, one can always choose a covering root datum $\tilde{E}$ such that $\tilde{\Pi}$ and $\tilde{\Pi}^\vee$ are $\mathbb{R}$-linearly independent.

To apply this to the study of rational functions $S_{x,w}$, $\tilde{S}_{x,w}$ associated to W on V and $\tilde{W}$ on $\tilde{V}$, for example, note that π induces an algebra homomorphism $S(\tilde{V}) \to S(V)$ which extends to a homomorphism of the localizations at multiplicative subsets generated by the roots, which we still denote as $\pi : S(\tilde{V})[\tilde{\Phi}^{-1}] \to S(V)[\Phi^{-1}]$. It is easy to see from the recurrence formulae (1.4.2) that (for Φ reduced) $\pi(\tilde{S}_{x,w}) = S_{\theta x, \theta w}$, so most results proved for $\tilde{S}_{x,w}$ for $\tilde{W}$ (for which the roots are linearly independent) immediately give corresponding facts for $S_{x,w}$.

Remarks. For non-reduced Φ, the BGG-Demazure operators x_w are still well-defined up to multiplication by a unit of any ring A over which Φ is defined. Thus, the rational functions $\hat{S}_{x,w}$ etc are only well-defined up to unit factors, so more care is needed in dealing with the "recurrence formulae" for them. Most results in Sections 1–2 extend to non-reduced root systems; the assumption that Φ is reduced is necessary only to avoid the more complicated statements in the non-reduced case.

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Department of Mathematics, 255 Hurley Building, University of Notre Dame, Notre Dame, Indiana, 46556-4618, U.S.A.

E-mail address: Dyer.1@nd.edu