LINEAR ALGEBRA CONSTRUCTION OF FORMAL KAZHDAN-LUSZTIG BASES

MATTHEW J. DYER

Abstract. General facts of linear algebra are used to give proofs for the (well-known) existence of analogs of Kazhdan-Lusztig polynomials corresponding to formal analogs of the Kazhdan-Lusztig involution, and of explicit formulae (some new, some known) for their coefficients in terms of coefficients of other natural families of polynomials (such as the corresponding formal analogs of the Kazhdan-Lusztig R-polynomials).

Introduction

In [13], Kazhdan and Lusztig associated to each pair of elements x, y of a Coxeter system a polynomial $P_{x,y} \in \mathbb{Z}[q]$. These Kazhdan-Lusztig polynomials and their variants (e.g g-polynomials of Eulerian lattices [23]) have a rich and interesting theory, with significant known or conjectured applications in representation theory, geometry and combinatorics. Many basic questions about them remain open in general e.g. the non-negativity of the coefficients of $P_{x,y}$ is known for crystallographic Coxeter systems by intersection cohomology techniques (see e.g. [14]) but not in general, though non-negativity of coefficients of g-vectors of face lattices of arbitrary (i.e. possibly non-rational) convex polytopes has been recently established (see [24], [21], [5], [1], [12]).

It is well known that formal analogs $\{p_{x,y}\}_{x,y\in\Omega}$ of the Kazhdan-Lusztig polynomials may be associated to any family of Laurent polynomials $r_{x,y} \in \mathbb{Z}[v, v^{-1}]$, for x, y elements of a poset Ω with finite intervals, satisfying suitable conditions abstracted from properties of the Kazhdan-Lusztig R-polynomials [23]. In view of the many important special cases or variants of this type of construction (see e.g [19], [18], [6], [17]) several essentially equivalent formalisms for it appear in the literature e.g. the incidence algebra formulation of Gabber [16] and Stanley [23] (cf 1.3), or formalisms involving analogs $\{t_w\}_{w\in W}$, $\{c'_w\}$ of the "standard" and "Kazhdan-Lusztig" (or "IC") bases of the generic Iwahori-Hecke algebra $\mathcal{H}$, such as [7], [16], [8] (cf 1.4).

In [8], there is a formula

$$(*) \qquad c'_w = (1-\rho_-\theta)^{-1}(1-\rho_-)(t_w) = \sum_{n\in\mathbb{N}} (\rho_-\theta)^n(t_w)$$

for the formal Kazhdan-Lusztig basis elements $\{c'_w := \sum_x p_{x,w} t_x\}_{w\in\Omega}$ in terms of the analog θ of the Kazhdan-Lusztig involution of $\mathcal{H}$, and a certain truncation operator ρ_- (see 1.4). This formula bears interesting comparison with (but does not correspond exactly to) Deligne's recursive construction of intersection cohomology (IC) complexes [2]. Taking coefficients of t_x in $(*)$ gives a formula for $p_{x,y}$ in terms

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of coefficients of the Laurent polynomials $r_{u,w}$ for $x \leq u \leq w \leq y$ which can be shown to be essentially equivalent to the main result of [4].

As observed in [8], $(*)$ above is a trivial consequence of the existence and uniqueness of the $p_{x,y}$ corresponding to the $r_{x,y}$. In this note, we instead give a different proof of $(*)$, by reducing it to the trivial case $\sharp(\Omega) = 1$, using only general facts of linear and homological algebra. We obtain as a consequence another proof of existence and uniqueness of the c'_y and hence the $p_{x,y}$. The main idea is to study the category of modules (over some fixed ring) equipped with an involution θ and an idempotent endomorphism ρ_- satisfying the identities $(1-\theta)(1-\rho_-\theta)^{-1}(1-\rho_-) = 0$ and $\ker(1+\theta) = \mathrm{Im}\,(1-\theta)$ which hold in the above situation, and show that it is an exact category (in the sense of Quillen [22]) with favorable stability properties for limits and filtered colimits required for infinite posets Ω.

The arrangement of this paper is s follows. Section 1 describes (without proofs) versions of the standard "incidence algebra" and "Kazhdan-Lusztig basis" formalisms for abstract Kazhdan-Lusztig polynomials $p_{x,y}$, and then briefly sketches (Proposition 1.5– 1.7) the "linear algebra" formalism of this paper. Section 2 describes some of the many equivalent reformulations of the identity $(1-\theta)(1-\rho\theta)^{-1}(1-\rho) = 0$ between otherwise arbitrary endomorphisms ρ, θ of some module. Section 3 observes that categories of additive functors preserving "exactness conditions" such as the above conditions $\ker(1+\theta) = \mathrm{Im}\,(1-\theta)$ and $\ker\rho = \mathrm{Im}\,(1-\rho)$ form exact categories. This principle is well known and widely used in special cases (e.g. in constructing exact categories of filtered modules); we give a version which is general enough for other applications to be given elsewhere. Section 4 specializes the generalities of Sections 2 and 3 to prove the main properties of the linear algebra formalism. Section 5 shows how the main points of the standard incidence algebra and basis formalisms follow by specialization of the linear algebra formalism. We take the opportunity to mention in a brief Section 6 some open conjectures involving Kazhdan-Lusztig polynomials and g-polynomials of Eulerian lattices.

We have used abelian categories as the general framework in this paper (e.g. so as to shorten certain arguments by use of duality) and assume that the reader is familiar with general category theory and the definition and basic properties of abelian and additive categories as can be found in [20] and [25] for example (by a pre-additive category, we mean in this note an ab-category in the sense of [20]). As suggested above, our results naturally lead us to consider exact categories, but for most purposes it is enough to regard them just as extension-closed subcategories of abelian (or module) categories, as discussed in 3.1. As a general point of notation, we use matrix notation for morphisms between direct sums: a morphism

$$\varphi\colon M_1 \oplus \ldots \oplus M_m \to N_1 \oplus \ldots \oplus N_n$$

is denoted by an $n \times m$ matrix (φ_{ij}) with $\varphi_{ij} \in \mathrm{Hom}(M_j, N_i)$, in an obvious way.

There are some representation-theoretic and combinatorial situations in which analogs of the $p_{x,y}$ are conjectured to exist but cannot be computed by the standard formalisms described above (since the analog θ of the Kazhdan-Lusztig involution is not an involution i.e. $r^{-1} \neq \overline{r}$ and no known analog of the IC degree conditions $p_{x,x} = 1$ and $p_{x,y} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $x \neq y$ holds). For instance, we will conjecture more precisely elsewhere that the theory [23] of relative g-vectors of Eulerian posets generalizes to a theory of considerable interest at least for graded Cohen-Macaulay lattices and that the analogs of the g-vectors (resp., relative g-vectors) describe multiplicities of simple modules in Verma modules (resp., analogs of principal series modules) for some graded highest weight representation categories defined (and dismissed prematurely as of little interest) in [10]. Although our main motivation to re-examine the formula $(*)$ was the hope that the more general formalism of Section 2, in which we assume neither that θ is an involution nor that ρ_- is idempotent, might be relevant to these problems, we have been unable to determine if there is an appropriate analog of ρ_-.

A sequel to this paper will be concerned with combinatorial consequences of and representation-theoretic explanations for existence of a generating function formula for $r_{x,y}$ (similar to that in [9]). Many of these do not require that $r^{-1} = \overline{r}$.

1. Formalisms for Kazhdan-Lusztig polynomials and bases

1.1. Let Ω be a poset. For $\Gamma \subseteq \Omega$, the set $\Lambda := \{\, y \in \Omega \mid y \leq z \text{ for some } z \in \Gamma \,\}$ is called a (poset) ideal of Ω; Λ is then said to be the ideal generated by Γ, and Λ is called a finitely generated ideal if it has some finite set of generators as ideal. The notions of a (finitely generated) coideal or a coideal generated by a subset of Ω are defined similarly, replacing $y \leq z$ by $y \geq z$.

For the remainder of this section, let Ω denote a poset in which all intervals $[x, y] := \{\, z \mid x \leq z \leq y \,\}$ are finite. Let $\mathcal{R} = \mathbb{Z}[v, v^{-1}]$ denote the ring of integral Laurent polynomials in the variable v.

1.2. **Incidence algebra formalism.** Recall that the incidence algebra $\mathcal{I}$ of Ω over $\mathcal{R}$ is defined as the set of all functions $f\colon \Omega \times \Omega \to \mathcal{R}$ such that $f(x, y) = 0$ unless $x \leq y$, equipped with pointwise addition and pointwise scalar multiplication by elements of $\mathcal{R}$, and convolution product

$$(fg)(x, y) := \sum_{z \in [x,y]} f(x, z)g(z, y).$$

For $f \in \mathcal{I}$, we write $f(x, y)$ as $f_{x,y}$. We say f is unitriangular if $f_{x,x} = 1$ for all $x \in \Omega$. We say f is IC-supported if f is unitriangular and $f_{x,y} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $x \neq y$ in Ω. A unitriangular (resp., IC-supported) element f is a unit in $\mathcal{I}$ and its inverse f^{-1} is also unitriangular (resp., IC-supported).

1.3. For any ring involution α of $\mathcal{R}$, there is a corresponding ring involution of $\mathcal{I}$, which we also denote in the same way as α, determined by $\big(\alpha(f)\big)_{x,y} = \alpha(f_{x,y})$. For the remainder of this section, we fix $\epsilon \in \{\pm 1\}$. Define the ring involution $a \mapsto \overline{a}$ (resp., $a \mapsto \hat{a}$) of $\mathcal{R}$ determined by $v \mapsto \epsilon v^{-1}$ (resp., $v \mapsto -v$). These determine ring involutions $f \mapsto \overline{f}$ (resp., $f \mapsto \hat{f}$) of $\mathcal{I}$. One has $\overline{\hat{a}} = \hat{\overline{a}}$ and similarly $\overline{\hat{f}} = \hat{\overline{f}}$. If f is unitriangular, one has of course $\overline{f^{-1}} = \overline{f}^{-1}$ and $\widehat{f^{-1}} = (\hat{f})^{-1}$.

Proposition. *Let r be a unitriangular element of $\mathcal{I}$ satisfying $r^{-1} = \overline{r}$.*

(a) *There are unique IC-supported elements $p, q \in \mathcal{I}$ satisfying $p = r\overline{p}$ and $q = \overline{q}r$.*
(b) *One has $\overline{r}^{-1} = r = \overline{\overline{r}}$, and p^{-1} (resp., q^{-1}) is the unique IC-supported element of $\mathcal{I}$ with $p^{-1} = \overline{p}^{-1}\overline{r}$ (resp., $q^{-1} = \overline{r}\,\overline{q}^{-1}$).*
(c) *If we define $s = q\overline{p}$, then s is unitriangular with $s = \overline{s}$.*
(d) *Suppose in* (a)–(c) *that $r = \hat{\overline{r}}$ i.e $r^{-1} = \overline{r} = \hat{r}$. Then $p^{-1} = \hat{q}$ and $s^{-1} = \hat{\overline{s}}$. Moreover, q (resp., p) is the unique IC-supported element of $\mathcal{I}$ with $q = s\hat{\overline{q}}$ (resp., $p = \hat{\overline{p}}s$).*

1.4. **Kazhdan-Lusztig basis formalism.** Now we indicate a reformulation of the above in terms of analogs of the Kazhdan-Lusztig bases of the Iwahori-Hecke algebra [13].

Suppose given Ω and $\{r_{x,y}\}_{x,y\in\Omega}$ as in 1.3(a). Define an $\mathcal{R}$-module $\mathcal{M}$ consisting of formal sums $m = \sum_{x\in\Omega} a_x t_x$ with $a_x \in \mathcal{R}$ such that the set of $x \in \Omega$ with $a_x \neq 0$ is contained in some finitely generated ideal of Ω. We regard $\mathcal{M}$ just as an abelian group. Then $\mathcal{M}$ has an involution θ given by

$$\theta\Big(\sum_{x\in\Omega} a_x t_x\Big) = \sum_{x\in\Omega}\Big(\sum_{y\in\Omega} r_{x,y}\overline{a}_y\Big)t_x.$$

There are also idempotent $\mathbb{Z}$-linear maps $\rho_\pm$ of $\mathcal{M}$ defined by

$$\rho_\pm\Big(\sum_{x\in\Omega} a_x t_x\Big) = \sum_x p_\pm(a_x)t_x$$

where each $a_x \in \mathcal{A}$ and $p_\pm(\sum_n b_n v^n) = \sum_{\pm n>0} b_n v^n$ for $b_n \in \mathbb{Z}$.

Proposition. (a) *There is for each $y \in \Omega$ a unique element c'_y (resp., c_y) in $\mathcal{M}$ satisfying $c'_y = \theta(c'_y)$ and $(1-\rho_-)(c'_y) = t_y$ (resp., $c_y = \theta(c_y)$ and $(1-\rho_+)(c_y) = t_y$).*

(b) *One has $c'_y = \sum_{x\in\Omega} p_{x,y}t_x$ (resp., $c_y = \sum_x (\overline{q}^{-1})_{x,y}t_x$) where p and q are as in* 1.3(a).

(c) $c'_y = \sum_{n\in\mathbb{N}}(\rho_-\theta)^n t_y$ *and* $c_y = \sum_{n\in\mathbb{N}}(\rho_+\theta)^n t_y$

(d) *One may naturally identify $\mathcal{M}$ with the set of all formal $\mathcal{R}$-linear combinations $\sum_x a_x c'_x$ (resp., $\sum_x a_x c_x$) such that the non-zero a_x are contained in some finitely generated ideal of Ω. Then, $c'_y = \sum_x s_{x,y}c_x$ where $s_{x,y}$ is as in* 1.3(c).

(e) *Define the involution $\theta'\colon \sum_x a_x t_x \mapsto \sum_x \hat{\overline{a}}_x t_x$ of $\mathcal{M}$ and the idempotent endomorphism $\rho'_\pm\colon \sum_x a_x c'_x \mapsto \sum_x p_\pm(a_x)c'_x$ of $\mathcal{M}$. Assume that $\overline{r} = \hat{r}$. Then $\theta\theta' = \theta'\theta$ and $\theta'(\sum_x a_x c'_x) = \sum_x \hat{\overline{a}}_x c_x$. Moreover, t_y is the unique element of $\mathcal{M}$ satisfying $t_y = \theta'(t_y)$ and $(1-\rho'_-)t_y = c'_y$. One has $t_y = \sum_{n\in\mathbb{N}}(\rho'_-\theta')^n c'_y$*

Remarks. If Ω is finitely generated as an ideal of itself (e.g. it is finite or has a minimum element), then $\{t_y\}_{y\in\Omega}$ $\{c_y\}_{y\in\Omega}$ and $\{c'_y\}_{y\in\Omega}$ are all $\mathcal{R}$-module bases of $\mathcal{M}$, and the formalism for c'_y is essentially equivalent the "IC-basis" formalism of [7]. In general, one may endow $\mathcal{M}$ with a natural topology (see 5.2) with respect to which all endomorphisms of (resp., sums in) $\mathcal{M}$ considered in this paper are continuous (resp., convergent).

1.5. **Linear algebra formalism.** Associate to any abelian category $\mathcal{A}$ the abelian category $\mathcal{A}'$ of triples $A = (M_A, \rho_A, \theta_A)$ where M_A is an object of $\mathcal{A}$ and ρ_A, θ_A are endomorphisms of M_A with ρ_A idempotent and θ_A an involution (see 4.1). Define the full additive subcategory $\mathcal{A}''$ of $\mathcal{A}'$ containing objects A of $\mathcal{A}'$ for which

$$\ker(1+\theta_A) = \operatorname{Im}(1-\theta_A) \tag{1.5.1}$$

$$(1-\theta_A)(1-\rho_A\theta_A)^{-1}(1-\rho_A) = 0 \tag{1.5.2}$$

where $1 = \mathrm{Id}_{M_A}$ (in particular, by assumption, $1-\rho_A\theta_A$ is invertible in $\mathrm{End}(M_A)$ and hence so is $1-\theta_A\rho_A$ by 2.1).

Proposition. (a) *For A in $\mathcal{A}''$, $(1-\rho_A\theta_A)^{-1}(1-\rho_A)$ and $(1-\theta_A\rho_A)^{-1}(1-\theta_A)$ (resp., $(1-\rho_A)(1-\theta_A\rho_A)^{-1}$ and $(1-\theta_A)(1-\rho_A\theta_A)^{-1}$) are orthogonal idempotents in* $\mathrm{End}(M_A)$ *summing to the identity. In particular, $M_A = \ker(1-\rho_A) \oplus \ker(1-\theta_A)$ and $M_A = \mathrm{Im}\,(1-\rho_A) \oplus \mathrm{Im}\,(1-\theta_A)$.*

(b) *The subcategory $\mathcal{A}''$ is closed under extensions in $\mathcal{A}'$ i.e. in any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}'$, if A and C are objects of $\mathcal{A}''$ then B is in $\mathcal{A}''$ also.*

(c) *Suppose that $\mathcal{A}$ is complete and has exact filtered colimits. Then $\mathcal{A}'$ is complete with exact filtered colimits and the subcategory $\mathcal{A}''$ is closed under formation of limits and filtered colimits in $\mathcal{A}'$.*

Remarks. In (a), $\ker(1-\theta_A) = \mathrm{Im}\,\big((1-\rho_A\theta_A)^{-1}(1-\rho_A)\big)$ (resp., $\ker(1+\theta_A) = \mathrm{Im}\,(1-\theta_A) = \ker\big((1-\rho_A)(1-\theta_A\rho_A)^{-1}\big)$) is the subobject of θ_A-invariants (resp., θ_A-anti-invariants) of M_A.

1.6. Now we consider objects M of $\mathcal{A}$ with several compatible structures as an object of $\mathcal{A}''$.

Proposition. *Suppose given an object M of $\mathcal{A}$, an involutory endomorphism θ of M and two orthogonal idempotent endomorphisms $\rho_\pm$ of M such that both triples $(M,\rho_\pm,\theta)$ are in $\mathcal{A}''$.*

(a) *There is a direct sum decomposition*

$$M = \mathrm{Im}\,(\rho_-) \oplus \mathrm{Im}\,\big((1-\theta\rho_-)^{-1}\rho_0\big) \oplus \theta\,\mathrm{Im}\,(\rho_-)$$

where $\rho_0 := (1-\rho_- - \rho_+)$.

(b) *Denote the idempotents in* $\mathrm{End}(M)$ *given by projections on the three direct summands of M in* (a) *as ρ'_-, ρ'_0 and ρ'_+ successively. Then $\theta\rho'_\pm = \rho'_\mp\theta$*

(c) *Suppose further that θ' is an involution of M commuting with θ and satisfying $\theta'\rho_\pm = \rho_\mp\theta'$ and $(1-\theta')(1-\rho'_-\theta')^{-1}(1-\rho') = 0$. Then both triples $(M,\rho'_\pm,\theta')$ are in $\mathcal{A}''$. Moreover there is a direct sum decomposition*

$$M = \mathrm{Im}\,(\rho'_-) \oplus \mathrm{Im}\,\big((1-\theta'\rho'_-)^{-1}\rho'_0\big) \oplus \theta'\,\mathrm{Im}\,(\rho'_-)$$

where $\rho'_0 := (1-\rho'_- - \rho'_+)$ and the idempotents in $\mathrm{End}(M)$ *corresponding to the three successive direct summands are ρ_-, ρ_0 and ρ_+.*

1.7. **Relations of the formalisms.** It is well known that if $\mathcal{A}$ is the (abelian) category of unitary R-modules over an associative ring R with identity then $\mathcal{A}$ is complete with exact filtered colimits. Taking $R = \mathbb{Z}$, define $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ as in 1.5. Let $\mathcal{M}$, θ, $\rho_\pm$, θ', $\rho'_\pm$ be as in 1.4.

Proposition. (a) *The quadruple $(\mathcal{M},\theta,\rho_+,\rho_-)$ attached to Ω satisfies the assumptions of* Proposition 1.6.

(b) *The idempotents $\rho'_\pm$ are those given by* 1.6(b).

(c) *If $\hat{r} = \overline{r}$, then $(\theta',\rho'_\pm)$ satisfy the conditions in* Proposition 1.6(c).

The Proposition follows directly from 1.5(a)–(b), and Propositions 1.3 and 1.4 can be deduced from it; see Section 5 for details.

2. Transverse endomorphisms

Many of the results of this section are widely known in the most important special cases, but for lack of a suitable systematic reference, we supply proofs.

2.1. Let $\mathcal{A}$ be an abelian category, M an object of $\mathcal{A}$, and $R := \mathrm{End}_{\mathcal{A}}(M)$. Then R is an associative ring with identity $1 := \mathrm{Id}_M$. We identify the opposite ring R^{op} with $\mathrm{End}_{\mathcal{A}^{\mathrm{op}}}(M^{\mathrm{op}})$ where $\mathcal{A}^{\mathrm{op}}$ is the opposite abelian category of $\mathcal{A}$ and M^{op} denotes M regarded as an object of $\mathcal{A}^{\mathrm{op}}$.

Fix idempotents e, f in R and elements $\rho \in eRf$, $\theta \in fRe$. The centralizer rings eRe and fRf are associative rings with identity elements e and f respectively. In our applications, we will have $e = f = 1$, and the reader may assume this below if desired.

Lemma. *The element $e - \rho\theta$ is a unit of eRe iff $f - \theta\rho$ is a unit of fRf, in which case their inverses (denoted by abuse of notation here as $(e-\rho\theta)^{-1}$, $(f-\theta\rho)^{-1}$) in those centralizer rings satisfy*

$$(e-\rho\theta)^{-1} = e + \rho(f-\theta\rho)^{-1}\theta, \qquad (f-\theta\rho)^{-1} = f + \theta(e-\rho\theta)^{-1}\rho.$$

Proof. Suppose that $e - \rho\theta$ is invertible in eRe. Then

$$(f-\theta\rho)\big(f+\theta(e-\rho\theta)^{-1}\rho\big) = f - \theta\rho + \theta(e-\rho\theta)(e-\rho\theta)^{-1}\rho = f$$

and similarly $\big(f+\theta(e-\rho\theta)^{-1}\rho\big)(f-\theta\rho) = f$. Since $f + \theta(e-\rho\theta)^{-1}\rho \in fRf$, this proves $(f-\theta\rho)^{-1} = f + \theta(e-\rho\theta)^{-1}\rho$, and the remainder of the lemma follows by symmetry. □

2.2. In the case $e = f = 1$, the equivalence of (i)–(iv) below was noted in [8].

Proposition. *Assume that $(e-\rho\theta)^{-1}$ and $(f-\theta\rho)^{-1}$ as above exist. Then the following conditions are equivalent:*

(i) $(e-\theta)(e-\rho\theta)^{-1}(e-\rho) = 1 - f$
(ii) $(f-\rho)(f-\theta\rho)^{-1}(f-\theta) = 1 - e$
(iii) $(e-\rho\theta)^{-1}(e-\rho) + (f-\theta\rho)^{-1}(f-\theta) = 1$
(iv) $(f-\rho)(f-\theta\rho)^{-1} + (e-\theta)(e-\rho\theta)^{-1} = 1$
(v) $(e-\rho\theta)^{-1}(e-\rho)$ *and* $(f-\theta\rho)^{-1}(f-\theta)$ *are orthogonal idempotents in R summing to* 1
(vi) $(f-\rho)(f-\theta\rho)^{-1}$ *and* $(e-\theta)(e-\rho\theta)^{-1}$ *are orthogonal idempotents in R summing to* 1
(vii) $\ker(f-\theta) \oplus \ker(e-\rho) \xrightarrow{\cong} M$, $\ker(f-\theta) \subseteq \mathrm{Im}\,(e)$, *and* $\ker(e-\rho) \subseteq \mathrm{Im}\,(f)$
(viii) $\big(\ker(f-\theta) \cap \mathrm{Im}\,(e)\big) + \big(\ker(e-\rho) \cap \mathrm{Im}\,(f)\big) = M$
(ix) $\mathrm{Im}\,(e-\theta) \oplus \mathrm{Im}\,(f-\rho) \xrightarrow{\cong} M$, $\ker(e) \subseteq \mathrm{Im}\,(f-\rho)$ *and* $\ker(f) \subseteq \mathrm{Im}\,(e-\theta)$
(x) $\big(\mathrm{Im}\,(e-\theta) + \ker(f)\big) \cap \big(\mathrm{Im}\,(f-\rho) + \ker(e)\big) = 0$
(xi) $\ker(e-\rho) = \mathrm{Im}\,\big((f-\theta\rho)^{-1}(f-\theta)\big)$ *and* $\mathrm{Im}\,\big(1-(f-\theta\rho)^{-1}(f-\theta)\big) \subseteq \mathrm{Im}\,(e)$
(xii) $(e-\rho)(f-\theta\rho)^{-1}(f-\theta) = 0$ *and* $\mathrm{Im}\,\big(1-(f-\theta\rho)^{-1}(f-\theta)\big) \subseteq \mathrm{Im}\,(e)$
(xiii) $\mathrm{Im}\,(e-\theta) = \ker\big((f-\rho)(f-\theta\rho)^{-1}\big)$ *and* $\ker\big(1-(f-\rho)(f-\theta\rho)^{-1}\big) \supseteq \ker(e)$
(xiv) $(f-\rho)(f-\theta\rho)^{-1}(e-\theta) = 0$ *and* $\ker\big(1-(f-\rho)(f-\theta\rho)^{-1}\big) \supseteq \ker(e)$
(xv) $(f-\theta)(e-\rho\theta)^{-1}(e-\rho) = 0 = (e-\rho)(f-\theta\rho)^{-1}(f-\theta)$ *and* $\ker(e) \cap \ker(f) = 0$
(xvi) $(e-\theta)(e-\rho\theta)^{-1}(f-\rho) = 0 = (f-\rho)(f-\theta\rho)^{-1}(e-\theta)$ *and* $\mathrm{Im}\,(e) + \mathrm{Im}\,(f) = M$

Remarks. By the symmetric (resp., dual) version of a condition or argument involving $(\mathcal{A}, M, R, e, f, \theta, \rho)$, we mean the one obtained by replacing $(\mathcal{A}, M, R, e, f, \theta, \rho)$ by $(\mathcal{A}, M, R, f, e, \rho, \theta)$ (resp., by $(\mathcal{A}^{\mathrm{op}}, M^{\mathrm{op}}, R^{\mathrm{op}}, f, e, \theta, \rho)$ and reformulating the resulting statements involving $\mathcal{A}^{\mathrm{op}}$, M^{op} and R^{op} as equivalent statements involving

$\mathcal{A}$, M and R). Since (i)–(iv) are (jointly) self-symmetric and self-dual, the symmetric, dual or dual symmetric versions of all of the conditions (v)–(xvi) could have been included above, but some of them have been omitted.

Proof. We begin with some preliminary remarks on some identifications, which we use subsequently without further comment, of some of the objects and maps appearing in the statement of the lemma. First, recall that for any idempotent $\alpha \in R$, one has

$$\ker \alpha = \mathrm{Im}\,(1-\alpha), \qquad \ker(1-\alpha) = \mathrm{Im}\,\alpha, \qquad \ker\alpha \oplus \mathrm{Im}\,\alpha \xrightarrow{\cong} M \tag{2.2.1}$$

(we use ker (etc) to denote either the kernel (etc) object or map, with usage distinguished by context). Further, if $\beta \in R$ is idempotent and $g \in \beta R \beta$ is invertible in $\beta R\beta$ with inverse in $\beta R \beta$ denoted as g^{-1}, then for any morphisms h (resp., k) in $\mathcal{A}$ with domain (resp., codomain) M and $h = h\beta$ (resp., $k = \beta k$), we have

$$\mathrm{Im}\,(hg) = \mathrm{Im}\,(h), \qquad \ker(gk) = \ker(k).$$

Now we prove that (i)–(iv) are equivalent. By Lemma 2.1, we have

$$\begin{aligned}
&(e-\rho\theta)^{-1}(e-\rho) + (f-\theta\rho)^{-1}(f-\theta)\\
=&\big(e+\rho(f-\theta\rho)^{-1}\theta\big)(e-\rho) + (f-\theta\rho)^{-1}(f-\theta)\\
=&e-\rho+\rho(f-\theta\rho)^{-1}\big((f-\theta\rho)-(f-\theta)\big) + (f-\theta\rho)^{-1}(f-\theta)\\
=&e+(f-\rho)(f-\theta\rho)^{-1}(f-\theta).
\end{aligned}$$

By the symmetric and dual arguments, the following four elements of R are equal:

$$\begin{aligned}
(e-\rho\theta)^{-1}(e-\rho) + (f-\theta\rho)^{-1}(f-\theta) &= e+(f-\rho)(f-\theta\rho)^{-1}(f-\theta)\\
= f+(e-\theta)(e-\rho\theta)^{-1}(e-\rho) &= (f-\rho)(f-\theta\rho)^{-1} + (e-\theta)(e-\rho\theta)^{-1}
\end{aligned}$$

Hence any one of the above four elements is equal to 1 iff they all are, as desired.

Now we record some general consequences of (i)–(iv). Multiplying (ii) (resp., (i)) on the left or right by e or $1-e$ (resp., by f or $1-f$) gives

$$\begin{aligned}
(e-\rho)(f-\theta\rho)^{-1}(f-\theta) = 0 &= (f-\rho)(f-\theta\rho)^{-1}(e-\theta)\\
(1-e)(f-\theta\rho)^{-1}(f-\theta) = 1-e &= (f-\rho)(f-\theta\rho)^{-1}(1-e)\\
(f-\theta)(e-\rho\theta)^{-1}(e-\rho) = 0 &= (e-\theta)(e-\rho\theta)^{-1}(f-\rho)\\
(1-f)(e-\rho\theta)^{-1}(e-\rho) = 1-f &= (e-\theta)(e-\rho\theta)^{-1}(1-f)
\end{aligned}$$

From (iii), (iv) we have

$$\begin{aligned}
M = \mathrm{Im}\,(1) &\subseteq \mathrm{Im}\,\big((e-\rho\theta)^{-1}(e-\rho)\big) + \mathrm{Im}\,\big((f-\theta\rho)^{-1}(f-\theta)\big) \subseteq \mathrm{Im}\,(e) + \mathrm{Im}\,(f)\\
0 = \ker(1) &\supseteq \ker\big((f-\rho)(f-\theta\rho)^{-1}\big) \cap \ker\big((e-\theta)(e-\rho\theta)^{-1}\big) \supseteq \ker(f)\cap\ker(e)
\end{aligned}$$

Note also that by (i)–(ii),

$$\begin{aligned}
\ker(f-\theta) \subseteq \ker(1-e) = \mathrm{Im}\,(e), &\qquad \ker(e-\rho) \subseteq \ker(1-f) = \mathrm{Im}\,(f)\\
\mathrm{Im}\,(e-\theta) \supseteq \mathrm{Im}\,(1-f) = \ker(f), &\qquad \mathrm{Im}\,(f-\rho) \supseteq \mathrm{Im}\,(1-e) = \ker(e).
\end{aligned}$$

We shall now assume that (i)–(iv) hold and show that the remaining conditions (v)–(xvi) follow. First, we show $\lambda := (e-\rho\theta)^{-1}(e-\rho)$ and $1-\lambda = (f-\theta\rho)^{-1}(f-\theta)$ are orthogonal idempotents in R. To prove this, it is enough to show that $(f-\theta)\lambda = 0$ (for then $(1-\lambda)\lambda = 0$). But by (i), $(e-\theta)\lambda = 1-f$ so

$$(f-\theta)\lambda = (f-\theta)e\lambda = f(e-\theta)\lambda = f(1-f) = 0.$$

Dually, $\lambda' := (f-\rho)(f-\theta\rho)^{-1}$ and $1-\lambda' = (e-\theta)(e-\rho\theta)^{-1}$ are orthogonal idempotents in R, so (v) and (vi) hold. By the above listed general consequences of (i)–(iv) and the remarks at the beginning of the proof, especially by (2.2.1) applied with $\alpha = \lambda$, $1-\lambda$, λ' or $1-\lambda'$, we see that (i)–(vi) imply the remaining conditions (vii)–(xvi).

It is clear that (v) (resp., (vi)) implies (iii) (resp., (iv)). To complete the proof, we shall now show that each of (vii)–(xvi) implies one of the conditions (i)–(vi). Note first that (vii) (resp., (ix), (xi), (xiii)) obviously implies (viii) (resp., (x), (xii), (xiv)) and that (viii) and (x) (resp., (xii) and (xiv), resp., (xv) and (xvi)) are duals or symmetric duals of one another. Hence it will be enough to show that (viii) (resp., (xii), (xv)) implies (iii) (resp., (ii), (i)).

Assume that (viii) holds i.e. $\begin{pmatrix} i_\theta & i_\rho \end{pmatrix} : K_\theta \oplus K_\rho \to M$ is an epimorphism where i_θ, i_ρ are the subobjects of M given by $i_\theta := \ker(f-\theta) \cap \mathrm{Im}\,(e) \colon K_\theta \to M$ and $i_\rho := \ker(e-\rho) \cap \mathrm{Im}\,(f) \colon K_\rho \to M$. Then

$$(e-\rho)\begin{pmatrix} i_\theta & i_\rho \end{pmatrix} = \begin{pmatrix} (e-\rho)i_\theta & 0 \end{pmatrix} = \begin{pmatrix} (e-\rho\theta)i_\theta & 0 \end{pmatrix}$$

since $i_\theta \subseteq \ker(f-\theta)$ gives $\rho i_\theta = \rho f i_\theta = \rho\theta i_\theta$. Thus,

$$(e-\rho\theta)^{-1}(e-\rho)\begin{pmatrix} i_\theta & i_\rho \end{pmatrix} = \begin{pmatrix} e i_\theta & 0 \end{pmatrix} = \begin{pmatrix} i_\theta & 0 \end{pmatrix}$$

since $i_\theta \subseteq \mathrm{Im}\,(e) = \ker(1-e)$ implies $e i_\theta = i_\theta$. Similarly,

$$(f-\theta\rho)^{-1}(f-\theta)\begin{pmatrix} i_\theta & i_\rho \end{pmatrix} = \begin{pmatrix} 0 & i_\rho \end{pmatrix}.$$

Adding gives $\left((e-\rho\theta)^{-1}(e-\rho) + (f-\theta\rho)^{-1}(f-\theta)\right)\begin{pmatrix} i_\theta & i_\rho \end{pmatrix} = \begin{pmatrix} i_\theta & i_\rho \end{pmatrix}$. Since $\begin{pmatrix} i_\theta & i_\rho \end{pmatrix}$ is an epimorphism, (iii) follows.

Now assume (xii) holds. Then $(e-\rho)(f-\theta\rho)^{-1}(f-\theta) = 0$, and since $\mathrm{Im}\,(e) = \ker(1-e)$, we get $(1-e)(f-\theta\rho)^{-1}(f-\theta) = (1-e)$. Also, $(f-1)(f-\theta\rho)^{-1}(f-\theta) = 0$ since $(f-1)f = 0$. Adding these three equations gives (ii).

Finally, assume (xv) holds. Define $\gamma := e - 1 + (f-\rho)(f-\theta\rho)^{-1}(f-\theta) = f - 1 + (e-\theta)(e-\rho\theta)^{-1}(e-\rho)$. Then $e\gamma = (e-\rho)(f-\theta\rho)^{-1}(f-\theta) = 0$ by (xv) and similarly $f\gamma = 0$. Hence $\mathrm{Im}\,(\gamma) \subseteq \ker(e) \cap \ker(f) = 0$ so $\gamma = 0$, and (i) holds. □

2.3. In this subsection, we assume that $e = f = 1$ in 2.2. Note there are significant simplifications in the statements of 2.2; in particular, all but the first subcondition in (vii)–(xvi) holds automatically. If the conditions of 2.2 hold in this case, we say that $\alpha := 1-\theta$ and $\beta := 1-\rho$ are transverse endomorphisms of M (or transverse elements of R). One has $1-\rho\theta = \alpha+\beta-\beta\alpha$ and $1-\theta\rho = \alpha+\beta-\alpha\beta$, so, for instance, α and β are transverse iff $\alpha+\beta-\alpha\beta$ is a unit of R and $\alpha(\alpha+\beta-\beta\alpha)^{-1}\beta = 0$. If α and β are transverse, then $M = \ker\alpha \oplus \ker\beta$ and $M = \mathrm{Im}\,\alpha \oplus \mathrm{Im}\,\beta$.

As a trivial example, if α and β are two orthogonal idempotents summing to 1, they are transverse.

3. Some exact categories of functors

In this section, it is terminologically convenient to use the notion of exact categories (in the sense of Quillen). As general references for exact categories, see [22] and [15]. Most of what we shall require, however, is described in the following subsection 3.1.

3.1. An exact category in the sense of Quillen is an additive category C with a given class E of diagrams of objects and maps in C of the form

$$0 \to A' \to A \to A'' \to 0 \tag{3.1.1}$$

subject to suitable axioms. Examples include any abelian category with the class of all its short exact sequences, and any additive category with the class of all its split short exact sequences. In general, the elements of E are called short exact sequences of C, and a functor between exact categories is called exact if it is additive and sends short exact sequences to short exact sequences.

A full additive subcategory C' of C is said to be closed under extensions if for any short exact sequence (3.1.1), if A' and A'' are in C' then A is in C'; then C' has a natural structure of exact category, with short exact sequences those short exact sequences of C whose objects are all in C'. Moreover, any small exact category is equivalent as exact category to an extension-closed subcategory of an abelian category, regarded as exact category in the above way.

3.2. Let C be a pre-additive category, and $Z \subseteq \mathbb{Z}$ be a subset with the property that $p \leq q \leq r$ in $\mathbb{Z}$ with $p, r \in Z$ implies $q \in Z$. By a Z-supported complex $A^\bullet$ in C, we mean a family of objects A^p of C for each $p \in Z$ with given maps $d_p \colon A^p \to A^{p+1}$ defined if $p, p+1 \in Z$ (called the differentials of $A^\bullet$) and satisfying $d_p d_{p-1} = 0$ if $p-1, p, p+1 \in Z$.

Assume henceforward that C is an exact category. We say $A^\bullet$ is exact if there are objects B^p for $p \in \mathbb{Z}$ with p or $p-1$ in Z and short exact sequences

$$0 \to Bp \xrightarrow{h_{p-1}} A^p \xrightarrow{g_p} B^{p+1} \to 0$$

of C for $p \in Z$ such that $d_p = h_p g_p$ if $p, p+1 \in Z$. A morphism $\alpha \colon A \to B$ in an exact category is called admissible if, when regarded as a complex supported on say $\{0, 1\}$, it is exact. This implies in particular that α has a kernel, cokernel, image (kernel of its cokernel) and coimage (cokernel of its kernel).

Remarks. The results of Section 2 extend mutatis mutandis to the situation that $\mathcal{A}$ is assumed to be an exact category (instead of an abelian category). For example, consider Proposition 2.2 with, for simplicity, $e = f = 1$. To ensure validity of this Proposition in case $\mathcal{A}$ is merely assumed to be an exact category, it suffices to add the hypothesis that $1-\rho$, $1-\theta$ are admissible morphisms of $\mathcal{A}$, and assume that the sums, intersections and inclusions of (admissible) subobjects of M involved in the statements are admissible in a suitable sense. For example, one interprets $\ker(1-\rho) + \ker(1-\theta) = M$ (resp., $\mathrm{Im}\,(1-\rho) \cap \mathrm{Im}\,(1-\theta) = 0$) as the condition that the natural map $\ker(1-\rho) \oplus \ker(1-\theta) \to M$ is an admissible morphism with cokernel 0 (resp., $\mathrm{Im}\,(1-\rho) \oplus \mathrm{Im}\,(1-\theta) \to M$ is an admissible morphism with kernel 0).

3.3. Let R be any small pre-additive category and $\mathcal{A}$ be any exact category. The additive category $\mathcal{A}^R$ of additive functors $R \to \mathcal{A}$ has a natural structure of additive category. Define a short exact sequence in $\mathcal{A}^R$ to be a complex

$$0 \to F \to G \to H \to 0$$

in $\mathcal{A}^R$ which is pointwise exact i.e. when applied to each object of R, it yields a short exact sequence in $\mathcal{A}$.

Suppose given a family X of complexes of objects of R, which are supported on (possibly differing) subsets of $\mathbb{Z}$ of cardinality three or greater. Write $\mathcal{A}^R(X)$ for the full additive subcategory of $\mathcal{A}^R$ consisting of functors $F\colon R \to \mathcal{A}$ such that for each complex $P^\bullet$ in X, $F(P^\bullet)$ is exact in $\mathcal{A}$.

Lemma. (a) *When endowed with the class of short exact sequences defined above, $\mathcal{A}^R$ is an exact category.*

(b) *$\mathcal{A}^R(X)$ is a full additive subcategory, closed under extensions in $\mathcal{A}^R$, and so has a natural structure of exact category.*

(c) *Let $P^\bullet$ be a complex in X and γ be one of the differentials of $P^\bullet$. Let f denote either* ker, coker , Im *or* coim . *Then for any functor F in $A^R(X)$, $F(\gamma)$ is an admissible morphism of $\mathcal{A}$ and the functor*

$$F \mapsto f(F(\gamma))\colon \mathcal{A}^R(X) \to \mathcal{A}$$

is exact.

Remarks. There would be no loss of generality (but some loss of convenience in applications) in requiring all complexes in X to be supported on $\{-1, 0, 1\}$.

Proof. We give a proof in case $\mathcal{A}$ is an extension closed, full additive subcategory of an abelian category $\mathcal{B}$ (the general case can be reduced to this by using [15]). We assume without loss of generality that X consists of complexes all supported on $Z := \{-1, 0, 1\}$.

Note first that $\mathcal{B}^R$ is an abelian category, with finite direct sums, kernels and cokernels determined pointwise. The definitions show immediately that $\mathcal{A}^R$ is a full additive subcategory, closed under extensions, of $\mathcal{B}^R$ and that its short exact sequences as defined above are exactly the short exact sequences in $\mathcal{B}^R$ of objects of $\mathcal{A}^R$. This implies (a).

Now consider a short exact sequence $0 \to F \to G \to H \to 0$ of $\mathcal{A}^R$ with F and H in $\mathcal{A}^R(X)$. Fix a complex $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in X. We extend this in the (unique up to isomorphism) way to a cochain complex $D^\bullet$ supported on $\mathbb{Z}$, with $D^i = 0$ for $i \notin Z$ and $D^{-1} \to D^0 \to D^1$ equal to our fixed complex $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in X. Then

$$0 \to F(D^\bullet) \to G(D^\bullet) \to H(D^\bullet) \to 0$$

is a short exact sequence of cochain complexes in the abelian category $\mathcal{B}$, and its cohomology long exact sequence gives an exact sequence in $\mathcal{B}$

$$\begin{aligned} 0 \to \ker F(\alpha) \to \ker G(\alpha) \to \ker H(\alpha) \to& \\ \ker F(\beta)/\operatorname{Im} F(\alpha) \to \ker G(\beta)/\operatorname{Im} G(\alpha) \to \ker H(\beta)/\operatorname{Im} H(\alpha) \to& \\ \operatorname{coker} F(\beta) \to \operatorname{coker} G(\beta) \to \ker H(\beta) \to 0.& \end{aligned}$$

Since F and H are in $\mathcal{A}^R(X)$, we get

$$0 = \ker F(\beta)/\operatorname{Im} F(\alpha) = \ker H(\beta)/\operatorname{Im} H(\alpha) = \ker G(\beta)/\operatorname{Im} G(\alpha). \tag{3.3.1}$$

The long exact sequence also shows that the first (resp., fourth) sequences in

$$0 \to \ker F(\alpha) \to \ker G(\alpha) \to \ker H(\alpha) \to 0$$
$$0 \to F(A) \to G(A) \to H(A) \to 0$$
$$0 \to \operatorname{coim} F(\alpha) \to \operatorname{coim} G(\alpha) \to \operatorname{coim} H(\alpha) \to 0$$
$$0 \to \operatorname{coker} F(\beta) \to \operatorname{coker} G(\beta) \to \ker H(\beta) \to 0$$
$$0 \to F(C) \to G(C) \to H(C) \to 0$$
$$0 \to \operatorname{Im} F(\beta) \to \operatorname{Im} G(\beta) \to \operatorname{Im} H(\beta) \to 0$$

are exact. The second (resp., fifth sequences) above are exact by the definitions, and the third (resp., sixth) sequences are therefore exact by application of the 3×3-lemma. Using (3.3.1), the above sequences remain exact with α and β interchanged. We denote the resulting eight short exact sequences involving α or β collectively as $(*)$. In a short exact sequence in $(*)$, all terms except perhaps the middle one are in $\mathcal{A}$. Since $\mathcal{A}$ is closed under extensions in $\mathcal{B}$, the middle term of each sequence in $(*)$ is also in $\mathcal{C}$ and the sequences in $(*)$ are (by definition) short exact sequences of $\mathcal{A}$. Now considering the short exact sequences

$$\begin{gathered}0 \to \ker G(\alpha) \to G(A) \to \operatorname{coim} G(\alpha) \to 0\\ 0 \to \operatorname{Im} G(\alpha) \to G(B) \to \operatorname{coker} G(\alpha) \to 0\\ 0 \to \ker G(\beta) \to G(B) \to \operatorname{coim} G(\beta) \to 0\\ 0 \to \operatorname{Im} G(\beta) \to G(C) \to \operatorname{coker} G(\beta) \to 0\end{gathered}$$

(the second and third of which are canonically isomorphic) shows that G is in $\mathcal{A}^R(X)$, proving (b). The exactness of the sequences in $(*)$ now implies (c). □

3.4. We record the following consequence of 3.3 which is used in Section 4.

Lemma. (a) *Let $\gamma\colon A \to B$ be a morphism in R. Then the full additive subcategory of $\mathcal{A}_R(X)$ consisting of additive functors $F\colon R \to \mathcal{A}$ with $F(\gamma)$ an isomorphism (resp., admissible monomorphism, admissible epimorphism) is closed under extensions in $\mathcal{A}_R(X)$.*

(b) *Fix an object B of R. For $i = 1,\dots,n$, let γ_i be a morphism in R with domain or codomain B, let f_i denote either* $\ker$ *or* Im *correspondingly, and for each F in $\mathcal{A}_R(X)$, write the kernel or image of $F(\gamma_i)$ correspondingly as $f_i(F(\gamma_i))\colon M_{F,i} \to F(B)$. Then the full additive subcategory of $\mathcal{A}_R(X)$ consisting of additive functors $F\colon R \to \mathcal{A}$ with*

$$(f_1(F(\gamma_1)),\dots,f_n(F(\gamma_n)))\colon M_{F,1}\oplus\dots\oplus M_{F,n} \to F(B)$$

an isomorphism is closed under extensions in $\mathcal{A}_R(X)$.

Proof. Part (a) follows since the category in question is $\mathcal{A}_R(X')$ where X' is obtained by adjoining to X an additional complex $0 \to A \xrightarrow{\gamma} B$ or $A \xrightarrow{\gamma} B \to 0$ or both (it also follows from the five lemma). Part (b) follows by a simple application of the five lemma and (c) of the previous lemma. □

4. Linear algebra formalism

4.1. Let $\mathcal{A}$ be an abelian (or exact) category. We consider a category $\mathcal{A}'$ of triples $A = (M_A, \theta_A, \rho_A)$ where M_A is an object of $\mathcal{A}$ and θ_A, ρ_A are respectively an involution and an idempotent endomorphism of M_A. A morphism $A \to B$ in $\mathcal{A}'$ is by definition a morphism $f\colon M_A \to M_B$ in $\mathcal{A}$ such that $f\rho_A = \rho_B f$ and $f\theta_A = \theta_B f$. Composition of morphisms in $\mathcal{A}'$ is given by composition of corresponding morphisms in $\mathcal{A}$. A sequence $0 \to A \to B \to C \to 0$ is called exact in $\mathcal{A}'$ if $0 \to M_A \to M_B \to M_C \to 0$ is exact in $\mathcal{A}$.

It is easy to see that if $\mathcal{A}$ is an abelian (resp., exact) category then $\mathcal{A}'$ is an abelian (resp., exact) category also. If $\mathcal{A}$ is the category R-mod of unitary R-modules, for a unital ring R, then $\mathcal{A}'$ may be identified with the category of unitary R'-modules for some unital ring R' generated by R and additional elements ρ, θ subject to suitable relations.

Let $\mathcal{A}''$ denote the full additive subcategory of $\mathcal{A}'$ with objects the triples A in $\mathcal{A}'$ such that the following conditions hold:

(i) $1-\theta_A$ and $1-\rho_A = 0$ are transverse endomorphisms of M_A i.e. in $\mathrm{End}(M_A)$, $(1-\theta_A)(1-\rho_A\theta_A)^{-1}(1-\rho_A) = 0$ where $1 := \mathrm{Id}_{M_A}$

(ii) the complexes $M_A \xrightarrow{1-\theta_A} M_A \xrightarrow{1+\theta_A}$ and $M_A \xrightarrow{1-\rho_A} M_A \xrightarrow{\rho_A} M_A$ are exact.

In particular, (i) above means that $1-\rho_A\theta_A$ and $1-\theta_A\rho_A$ are automorphisms of M_A (by 2.1), and that ρ_A, θ_A, M_A satisfy the equivalent conditions on ρ, θ, M in 2.2 (with $e = f = 1$).

Remarks. If $\mathcal{A}$ is abelian, the requirement that $M_A \xrightarrow{1-\rho_A} M_A \xrightarrow{\rho_A} M_A$ is exact is redundant (since it is just equivalent to $\rho_A^2 = \rho_A$), but for a general exact category it implies that ρ_A, $1-\rho_A$ are split idempotents (i.e. they have kernels, images, etc in $\mathcal{A}$).

4.2. As an application of 3.3, we have the following fact.

Lemma. *The category $\mathcal{A}''$ is closed under extensions in $\mathcal{A}'$. Hence $\mathcal{A}''$ has a natural structure of exact category.*

Proof. Let R' be the associative unital ring generated by generators ρ, θ and relations $\theta^2 = 1$ and $\rho^2 = \rho$. We form a pre-additive category R with one object, denoted $\bullet$, and $\mathrm{End}_R(\bullet) = R'$. Let X be the two-element set with elements the complexes $\bullet \xrightarrow{1-\theta} \bullet \xrightarrow{1+\theta} \bullet$ and $\bullet \xrightarrow{1-\rho} \bullet \xrightarrow{\rho} \bullet$. It is clear that $\mathcal{A}^R$ identifies with $\mathcal{A}'$; the identification takes the functor F to a triple $(F(\bullet), F(\theta), F(\rho)))$ in such a way that its (extension-closed by 3.3) full additive subcategory $\mathcal{A}^R(X)$ identifies with the full additive subcategory of $\mathcal{A}'$ consisting of triples A satisfying 4.1(ii) above. Now since 4.1(ii) implies that $1-\rho_A$ and $1-\theta_A$ are admissible morphisms, the condition 4.1(i) defining $\mathcal{A}''$ may by 2.2 be expressed as the conditions that $1-\rho_A\theta_A$ is an isomorphism and $\ker(1-\rho_A) \oplus \ker(1-\theta_A) \xrightarrow{\cong} M_A$ is an isomorphism. Imposing these conditions gives an extension-closed subcategory of $\mathcal{A}'$ by 3.4(a)–(b) and the equivalence $\mathcal{A}' \cong \mathcal{A}^R(X)$. □

4.3. Suppose henceforward that $\mathcal{A}$ is an abelian category.

Lemma. *If $\mathcal{A}$ is complete (resp., has exact filtered colimits) then $\mathcal{A}'$ is complete (resp., has exact filtered colimits) and $\mathcal{A}''$ is closed under limits (resp., filtered colimits) in $\mathcal{A}'$.*

Proof. Consider a functor $F\colon J \to \mathcal{A}'$; for an object α of J, we write $F(\alpha) = (M_\alpha, \theta_\alpha, \rho_\alpha)$. For a morphism $f\colon \alpha \to \beta$ $F(f)\colon F(\alpha) \to F(\beta)$ is a morphism $M_\alpha \to M_\beta$ in $\mathcal{A}$ which we write also as $F(f)$. One sees at once that forgetful functor $G\colon \mathcal{A}' \to \mathcal{A}$ creates limits i.e. if the composite $GF\colon J \to \mathcal{A}$ has a limit $M = \lim_\alpha M_\alpha$ with canonical maps $\pi_\alpha\colon M \to GF(\alpha) = M_\alpha$, then F has a limit $(M, \rho := \lim_\alpha \rho_\alpha, \theta := \lim_\alpha \theta_\alpha)$ with canonical maps determined by the π_α. Similarly, G creates colimits. Now if $\mathcal{A}$ has exact filtered limits, it is clear from the above that $\mathcal{A}'$ does also; since filtered colimits then commute with homology in $\mathcal{A}'$, the definition of $\mathcal{A}''$ implies it is closed under filtered colimits in $\mathcal{A}'$.

Now we consider closure of $\mathcal{A}''$ under limits in $\mathcal{A}'$. Consider a functor F as above with each F_α in $\mathcal{A}''$ and suppose $\lim GF$ exists in $\mathcal{A}$ (and so $\lim F = (M, \rho, \theta)$

exists in $\mathcal{A}'$ with canonical maps π_α as above). Clearly in $\mathrm{End}_{\mathcal{A}}(M)$, we have $\rho^2 = \rho$, $\theta^2 = 1 := \mathrm{Id}_M$, $(1-\rho\theta)$ is invertible and $(1-\theta)(1-\rho\theta)^{-1}(1-\rho) = 0$. Hence we need only show that $\ker(1+\theta) \subseteq \mathrm{Im}\,(1-\theta)$ as subobjects of M. Suppose that $h\colon L \to M$ is s monomorphism satisfying $(1+\theta)h = 0$. For an object α of J, write 1_α for Id_{M_α}. Then $(1_\alpha + \theta_\alpha)\pi_\alpha h = \pi_\alpha(1+\theta)h = 0$ so $\mathrm{Im}\,(\pi_\alpha h) \subseteq \ker(1_\alpha + \theta_\alpha) = \mathrm{Im}\,(1_\alpha - \theta_\alpha) = \ker\big((1_\alpha - \rho_\alpha)(1_\alpha - \theta_\alpha\rho_\alpha)^{-1}\big)$ by 2.2. Hence $\pi_\alpha(1-\rho)(1-\theta\rho)^{-1}h = (1_\alpha - \rho_\alpha)(1_\alpha - \theta_\alpha\rho_\alpha)^{-1}\pi_\alpha h = 0$ for all α and hence $h \subseteq \ker\big((1-\rho)(1-\theta\rho)^{-1}\big) = \mathrm{Im}\,(1-\theta)$ by 2.2 again as required.

□

4.4. In this subsection, fix a quadruple $(M, \theta, \rho_+, \rho_-)$ where M is an object of $\mathcal{A}$, θ is an involutory endomorphism of M and $\rho_\pm$ are orthogonal idempotent endomorphisms of M such that both triples $(M, \theta, \rho_\pm)$ are in $\mathcal{A}''$.

Lemma. (a) $\mathrm{Im}\,(1-\rho_+) = \mathrm{Im}\,(\rho_0) \oplus \mathrm{Im}\,(\rho_-) = \mathrm{Im}\,(\rho_-) \oplus (1-\rho_-\theta)^{-1}\mathrm{Im}\,(\rho_0)$

(b) $\mathrm{Im}\,(1-\rho_+) \cap \theta\,\mathrm{Im}\,(1-\rho_+) = \mathrm{Im}\,(1-\rho_+) \cap \ker(1-\theta) = (1-\rho_-\theta)^{-1}\mathrm{Im}\,(\rho_0)$

(c) *The morphism* $\mathrm{coim}\,(\rho_+)\theta\mathrm{Im}\,(\rho_-)$ *is an isomorphism.*

(d) $M = \mathrm{Im}\,(\rho_-) \oplus \theta\,\mathrm{Im}\,(\rho_-) \oplus (1-\rho_-\theta)^{-1}\mathrm{Im}\,(\rho_0)$.

Proof. The first equality in (a) is clear. For the second, note that

$$\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_0\big) = \mathrm{Im}\,\Big(\big(1+\rho_-(1-\theta\rho_-)^{-1}\theta\big)\rho_0\Big) \subseteq \mathrm{Im}\,(\rho_0) + \mathrm{Im}\,(\rho_-)$$

and

$$\mathrm{Im}\,(\rho_0) = \mathrm{Im}\,\big((1-\rho_-\theta)(1-\rho_-\theta)^{-1}\rho_0\big) \subseteq \mathrm{Im}\,\big(1-\rho_-\theta)^{-1}\rho_0\big) + \mathrm{Im}\,(\rho_-)$$

which implies that the second and third subobjects in (a) are included in one another and are hence equivalent. Now let $L = \mathrm{Im}\,(1-\rho_+) \cap \theta\,\mathrm{Im}\,(1-\rho_+)$. Then

$$\begin{aligned}\mathrm{Im}\,\big((1-\theta)L\big) &\subseteq \mathrm{Im}\,(1-\theta) \cap \big(\mathrm{Im}\,(1-\rho_+) + \theta^2\,\mathrm{Im}\,(1-\rho_+)\big)\\ &\subseteq \mathrm{Im}\,(1-\theta) \cap \mathrm{Im}\,(1-\rho_+) = 0\end{aligned}$$

so $L \subseteq \mathrm{Im}\,(1-\rho_+) \cap \ker(1-\theta) \subseteq L$ and the first equality in (b) holds. By (a), $\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_0\big) \subseteq \mathrm{Im}\,(1-\rho_+)$, so by 2.2 and the modular law,

$$\begin{aligned}L &= \mathrm{Im}\,\big((1-\rho_-\theta)^{-1}(1-\rho_-)\big) \cap \mathrm{Im}\,(1-\rho_+)\\ &= \Big(\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_0\big) + \mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_+\big)\Big) \cap \mathrm{Im}\,(1-\rho_+)\\ &= \mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_0\big) + \Big(\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_+\big) \cap \mathrm{Im}\,(1-\rho_+)\Big)\end{aligned}$$

Now (b) follows since

$$\begin{aligned}\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_+\big) \cap \mathrm{Im}\,(1-\rho_+) &= (1-\rho_-\theta)^{-1}\Big(\mathrm{Im}\,(\rho_+) \cap (1-\rho_-\theta)\mathrm{Im}\,(1-\rho_+)\Big)\\ &\subseteq (1-\rho_-\theta)^{-1}\Big(\mathrm{Im}\,(\rho_+) \cap \big(\mathrm{Im}\,(1-\rho_+) + \mathrm{Im}\,(\rho_-)\big)\Big) = 0.\end{aligned}$$

Write the image of $1-\theta$ as $\mathrm{Im}\,(1-\theta)\colon K \to M$. Then we have $\mathrm{Im}\,(\rho_\pm) = (1-\theta\rho_\pm)^{-1}\mathrm{Im}\,(1-\theta)$, and so we may (and do) canonically regard $\mathrm{Im}\,(1-\theta)$ and

$\mathrm{Im}\,(\rho_\pm)$ (resp., $\mathrm{coim}\,(1-\theta)$ and $\mathrm{coim}\,(\rho_\pm)$) as morphisms $K \to M$ (resp., $M \to K$). Since $\mathrm{Im}\,\rho_- = \rho_- \mathrm{Im}\,\rho_-$ and $\mathrm{coim}\,\rho_+ = \mathrm{coim}\,(\rho_+)\rho_+$, we get

$$\begin{aligned}\mathrm{coim}\,(\rho_+)\,\theta\,\mathrm{Im}\,(\rho_-) &= -\mathrm{coim}\,(\rho_+)\rho_+(1-\theta\rho_-)\rho_-\mathrm{Im}\,(\rho_-)\\ &= -\mathrm{coim}\,(\rho_+)(1-\theta\rho_+)(1-\theta\rho_+)^{-1}(1-\theta\rho_-)\mathrm{Im}\,(\rho_-)\\ &= -\mathrm{coim}\,(\rho_+)(1-\theta\rho_+)(1-\theta\rho_+)^{-1}\mathrm{Im}\,(1-\theta)\\ &= -\mathrm{coim}\,(\rho_+)(1-\theta\rho_+)\mathrm{Im}\,(\rho_+).\end{aligned}$$

Since $\rho_+(1-\theta\rho_+)^{-1} = \rho_+\big(1+\theta(1-\rho_+\theta)^{-1}\rho_+\big) = \rho_+(1-\theta\rho_+)^{-1}\rho_+$ and also $\rho_+(1-\theta\rho_+) = \rho_+(1-\theta\rho_+)\rho_+$, we get

$$\rho_+(1-\theta\rho_+)^{\pm 1}\rho_+(1-\theta\rho_+)^{\mp 1} = \rho_+$$

This implies that $\mathrm{coim}\,(\rho_+)(1-\theta\rho_+)^{\pm 1}\mathrm{Im}\,(\rho_+)$ are mutually inverse automorphisms of K, which proves (c) from above.

To prove (d), note first that (c) implies that $M = \mathrm{Im}\,(1-\rho_+) + \theta\,\mathrm{Im}\,(\rho_-)$. By (a), we then get $M = \mathrm{Im}\,(\rho_-) + (1-\rho_-\theta)^{-1}\mathrm{Im}\,(\rho_0) + \theta\,\mathrm{Im}\,(\rho_-)$. Using (b), (a) and θ applied to (a), one readily sees that this sum of subobjects of M is direct, giving (d). $\square$

Remarks. Since $(1-\theta\rho_\pm)\mathrm{Im}\,(\rho_\pm) = (1-\theta)\mathrm{Im}\,(\rho_\pm)$, we get also from the proof of (c) that $(1-\theta\rho_\pm)^{-1}(1-\theta)\mathrm{Im}\,(\rho_\mp) = \mathrm{Im}\,(\rho_\pm)$. Similarly, $(1-\rho_\pm)(1-\rho_\mp\theta)^{-1}\ker(\rho_\mp) = \ker(\rho_\pm)$.

4.5. Let P and Q be objects of an abelian category $\mathcal{A}$ such that $2\mathrm{Id}_Q\colon Q \to Q$ is a monomorphism. Let M' be an object of $\mathcal{A}$ with direct sum decomposition $M' = P \oplus Q \oplus P$. Suppose that $\alpha\colon Q \to P$, $\beta\colon P \to Q$ and $\gamma\colon P \to P$ are morphisms such that $\mathrm{Id}_P - \gamma$ and $\mathrm{Id}_P - \alpha\beta + \gamma$ are both automorphisms of P. Consider the following endomorphisms of M:

$$\theta = \begin{pmatrix} 0 & 0 & \mathrm{Id}_P\\ 0 & \mathrm{Id}_Q & 0\\ \mathrm{Id}_P & 0 & 0\end{pmatrix},\quad \rho_- = \begin{pmatrix}\mathrm{Id}_P & -\alpha & \alpha\beta-\gamma\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \rho_+ = \begin{pmatrix} 0 & 0 & \gamma\\ 0 & 0 & \beta\\ 0 & 0 & \mathrm{Id}_P\end{pmatrix},$$

$$\rho'_- = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \rho'_+ = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Lemma. (a) *The quadruple $(M', \theta, \rho_+, \rho_-)$ associated above to $(P, Q, \alpha, \beta, \gamma)$ satisfies the conditions of subsection* 4.4.

(b) *An endomorphism θ'' of M satisfies $\theta''\theta = \theta\theta''$, $\rho_-\theta'' = \theta''\rho_-$ and $\rho_+\theta'' = \theta''\rho_+$ iff it is of the form*

$$\theta'' = \begin{pmatrix} a & 0 & 0\\ 0 & e & 0\\ 0 & 0 & a\end{pmatrix}$$

where $a\colon P \to P$ and $e\colon Q \to Q$ satisfy $\alpha e = a\alpha$, $\beta a = e\beta$, $a\gamma = \gamma a$.

(c) *An endomorphism θ' of $M = P \oplus Q \oplus P$ satisfies $\theta\theta' = \theta'\theta$ and $\rho_+\theta' = \theta'\rho_-$ iff it is of the form*

$$\theta' = \begin{pmatrix}\gamma c & -c\alpha & c\\ \beta c & e & \beta c\\ c & -c\alpha & \gamma c\end{pmatrix}$$

where $c\colon P\to P$ *and* $e\colon Q\to Q$ *satisfy* $(\mathrm{Id}_P-\gamma)c=c(\mathrm{Id}_P-\alpha\beta+\gamma)$.

(d) *Consider an endomorphism* θ' *of* M *satisfying the conditions of* (b)*. If, further,* θ' *is an involution of* M *such that* $(1_M-\theta'\rho'_-)$ *is invertible and* $(1_M-\rho'_-)(1_M-\theta'\rho'_-)^{-1}(1_M-\theta')=0$ *then* $\ker(1+\theta')=\mathrm{Im}\,(1-\theta')$.

(e) *Any quadruple* (M,θ,ρ_-,ρ_+) *satisfying the conditions of* 4.4 *is isomorphic to a quadruple associated above to some* $(P,Q,\alpha,\beta,\gamma)$*, in such a way that the subobject* $P\oplus 0\oplus 0$ *(resp.,* $0\oplus 0\oplus P$*) of* $M=P\oplus Q\oplus P$ *identifies with* $\mathrm{Im}\,\rho_-$ *(resp.,* $\theta\,\mathrm{Im}\,\rho_-$*).*

Proof. The straightforward matrix computations required to prove (a)–(c) are omitted. For (d), one shows by similar computations that θ' as in (b) satisfies the additional assumptions of (d) iff $\big((1_P-\gamma)c\big)^2=1$ (so c is invertible with $c^{-1}=(1_P-\gamma)c(1_P-\gamma)$), $\big((1_P-\gamma c)c^{-1}\big)^2=1_P$, $e=1_Q-\beta c\alpha$, $(1_P-\gamma c+c)\alpha=0$ and $\beta(1_P-\gamma c+c)=0$. Then one can check that

$$\ker(1+\theta)=\mathrm{Im}\,(1-\theta)=\big(\,\gamma c{-}1\;\;\beta c\;\;c\,\big)^{\mathrm{t}}\colon P\to M'$$

where t denotes transpose, using that $2\mathrm{Id}_Q$ is a monomorphism.

To prove (e), fix a quadruple (M,θ,ρ_+,ρ_-) as in 4.4. Write $\mathrm{Im}\,\rho_-\colon P\to M$, $(1-\rho\theta)^{-1}\mathrm{Im}\,\rho_0)\colon Q\to M$ respectively and identify $P\oplus Q\oplus P\cong P\oplus Q\oplus\theta(P)\cong M$ using Lemma 4.4. That lemma implies the matrix of $\theta\colon P\oplus Q\oplus P\to P\oplus Q\oplus P$ is given by the above matrix for θ. Also, we have $M=\mathrm{Im}\,(\rho_-)\oplus\mathrm{Im}\,(\rho_0)\oplus\mathrm{Im}\,(\rho_+)$ and the matrix A of $\mathrm{Id}_M\colon P\oplus Q\oplus P\to\mathrm{Im}\,(\rho_-)\oplus\mathrm{Im}\,(\rho_0)\oplus\mathrm{Im}\,(\rho_+)$ is (invertible) upper triangular with $A_{11}=\mathrm{Id}_P$, $A_{22}=\mathrm{Id}_Q$ since $\rho_-\mathrm{Im}\,(\rho_-)=\mathrm{Im}\,(\rho_-)$, $\rho_+\mathrm{Im}\,(\rho_-)=0$, $\rho_0\mathrm{Im}\,(\rho_-)=0$, $\rho_+\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_0\big)=0$ and $\rho_0\mathrm{Im}\,\big((1-\rho_-\theta)^{-1}\rho_0\big)=\mathrm{Im}\,(\rho_0)$. The matrices of ρ_- and ρ_+ as endomorphisms of $P\oplus Q\oplus P$ are therefore given by $A^{-1}\begin{pmatrix}\mathrm{Id}_P&0&0\\0&0&0\\0&0&0\end{pmatrix}A$ and $A^{-1}\begin{pmatrix}0&0&0\\0&0&0\\0&0&\mathrm{Id}_P\end{pmatrix}A$ respectively, which implies they are of the forms given above with $\alpha=-A_{12}$, $\beta=-A_{23}$ and $\gamma=A_{12}A_{23}-A_{13}$. Using that $\mathrm{Id}-\rho_\pm\theta$ is invertible gives invertibility of $\mathrm{Id}_P-\gamma$ and $\mathrm{Id}_P-\alpha\beta+\gamma$, and exactness of $M\xrightarrow{1-\theta}M\xrightarrow{1+\theta}M$ implies that $2\mathrm{Id}_Q$ is a monomorphism. This completes the proof. $\square$

4.6. **Proof of Proposition 1.5.** Proposition 1.5(a) follows from Proposition 2.2, (with $e=f=1$). Also, 1.5(b) follows from 4.2 and 1.5(c) follows from 4.3.

4.7. **Proof of Proposition 1.6.** First, 1.6(a) follows from Lemma 4.4(d), and 1.6(b) follows immediately. For (c), one may assume without loss of generality by 4.5(e) that $M=M'$ where M' is as in 4.5. Then $\ker(1+\theta')=\mathrm{Im}\,(1-\theta')$ by 4.5(d). Hence (M,θ',ρ'_-) is in $\mathcal{A}''$. This implies that $(M,\theta\theta'\theta^{-1},\theta\rho'_-\theta^{-1},)=(M,\theta',\rho'_+)$ is in $\mathcal{A}''$ too. Then 1.6(c) follows from 4.5(e) and 1.6(a) applied to $(M,\theta',\rho'_\pm)$.

Remarks. Suppose that (M,θ,ρ_+,ρ_-) is as in 4.4. This data determines two projections $(1-\rho\theta_\pm)^{-1}(1-\rho_\pm)$ onto the subobject $\ker(1-\theta)$ of θ-invariants of M. If θ' is an automorphism of M satisfying the conditions of 4.5(b), then these two projections are conjugate by θ:

$$\theta'(1-\rho_-\theta)^{-1}(1-\rho_-)\theta'^{-1}=(1-\rho_+\theta)^{-1}(1-\rho_+)$$

Moreover, if θ' is an involution then

$$\theta'(1-\rho_-\theta)^{-1}\rho_0\theta'^{-1}=(1-\rho_+\theta)^{-1}\rho_0.$$

I do not know in general when such an (involutory) endomorphism θ' exists. However, if θ'_i for $i = 1, 2$ are any two such involutory automorphisms (i.e. satisfying the conditions of 4.5(b)), then $\theta'' := \theta'_1\theta'_2$ satisfies the conditions of 4.5(c).

5. Specialization of the linear algebra formalism

Throughout this section, we fix a poset Ω with finite intervals, an element $\epsilon \in \{\pm 1\}$, and a unitriangular element r of the incidence algebra $\mathcal{I}$ with $r^{-1} = \overline{r}$ where $h \mapsto \overline{h}$ is defined using ϵ as in 1.3. Set $\mathcal{A} = \mathbb{Z}$-mod and define $\mathcal{A}'$ and $\mathcal{A}''$ as in 1.5. Define $\mathcal{M}$ as in 1.3, regarding $\mathcal{M}$ as an object of $\mathcal{A}$ and regard θ, $\rho_\pm$, θ', $\rho'_\pm$ defined in 1.3 as endomorphisms of $\mathcal{M}$ as abelian group. Then $(\mathcal{M}, \theta, \rho_\pm)$ and $(\mathcal{M}, \theta', \rho'_\pm)$ are in $\mathcal{A}'$

5.1. **Proof of 1.7(a).** Clearly, ρ_- and ρ_+ are orthogonal idempotents. Because of the existence of the algebra automorphism of $\mathcal{A}$ given by $v^n \mapsto v^{-n}$, we need only prove that $(\mathcal{M}, \rho_-, \theta)$ is in $\mathcal{A}''$.

We say that $\Lambda \subseteq \Omega$ is convex in Ω if it is the intersection of an ideal and a coideal of Ω i.e. if $x, y \in \Lambda$, $z \in \Omega$ and $x \le z \le y$ imply $z \in \Lambda$. Denote the triple $(\mathcal{M}, \theta, \rho_-)$ in $\mathcal{A}'$ attached to the poset Ω and the family $\{r_{x,y}\}_{x,y\in\Omega}$ in $\mathcal{R}$ as $A_\Omega = (\mathcal{M}_\Omega, \theta_\Omega, \rho_\Omega)$. For any convex subset Γ of Ω, one has a triple A_Γ in $\mathcal{A}'$ defined similarly using Γ and $\{\, r_{x,y} \,\}_{x,y\in\Gamma}$. We wish to show that A_Ω is in $\mathcal{A}''$. Let I be the inclusion ordered set of finitely generated ideals of Ω; note I is directed by inclusion. There is an obvious direct system $\{A_\Gamma\}_{\Gamma\in I}$ of subobjects of A_Ω and essentially by definition $A_\Omega \cong \varinjlim_{\Gamma\in I} A_\Gamma$. Hence by 4.3 we need only check that A_Γ is in $\mathcal{A}''$ for fixed $\Gamma \in I$. Let J be the inclusion ordered family of finite coideals of Γ. We may regard the family $\{A_\Lambda\}_{\Lambda\in J}$ as an inverse system of quotient objects of A_Γ, and again essentially by definition, $A_\Gamma \cong \lim_{\Lambda\in J} A_\Gamma$. Hence by 4.3 again, we need only check that for a fixed finite $\Lambda \in J$, that A_Λ is in $\mathcal{A}''$. We prove this by induction on $n := \sharp(\Lambda)$. If $n \le 1$, the assertion follows by direct calculation. If $n \ge 2$, choose an ideal Σ of Λ with $\Sigma \ne \emptyset, \Lambda$. One may naturally regard A_Σ as a subobject of A_Γ, and there is a natural short exact sequence $0 \to A_\Sigma \to A_\Lambda \to A_{\Lambda\setminus\Sigma} \to 0$. By induction, A_Σ and $A_{\Lambda\setminus\Sigma}$ are in $\mathcal{A}''$ and hence A_Λ is in $\mathcal{A}''$ by 1.5(b).

5.2. For any finite subset Γ of Ω, let

$$U(\Gamma) := \{\, \sum_x a_x t_x \in \mathcal{M} \mid a_x = 0 \text{ if } x \ge y \text{ for some } y \in \Gamma \,\}.$$

We make $\mathcal{M}$ a (Hausdorff) topological abelian group with the sets $U(\Gamma)$ for finite subsets Γ of Ω as a basis of open sets containing 0. We call the topology so defined the standard topology on the set of formal sums $\sum_x a_x t_x$. The actions of $\rho_\pm$ and θ on $\mathcal{M}$ are continuous with respect to this topology. Observe that for any $m \in \mathcal{M}$, the series $\sum_{n\in\mathbb{N}} (\rho_\pm\theta)^n(m)$ is convergent in this topology, necessarily to $(1 - \rho_\pm\theta)^{-1}(m)$; this is clear from the description in 5.1 of A_Ω as a filtered colimit of limits of fiterated extensions of objects $A_{\{z\}}$ for $z \in \Omega$ and the fact $(\rho_\pm\theta)^2 = 0$ on $\mathcal{M}_{\{z\}}$. Similarly $\sum_{n\in\mathbb{N}} (\theta\rho_\pm)^n(m)$ is convergent to $(1 - \theta\rho_\pm)^{-1}(m)$.

Given any unitriangular element m of the incidence algebra $\mathcal{I}$, define the elements $n_y := \sum_{x\le y} m_{x,y} t_x$ of $\mathcal{M}$ for $y \in \Omega$. For any element $\sum_{x\in\Omega} a_x t_x$ in $\mathcal{M}$, the sum $\sum_x a_x n_x$ is convergent, and the map $\mathcal{M} \to \mathcal{M}$ defined by $\sum_{x\in\Omega} a_x t_x \mapsto \sum_{x\in\Omega} a_x n_x$ is a automorphism of $\mathcal{M}$ as topological abelian group. In particular, we may naturally identify $\mathcal{M}$ with the set of formal sums $\sum_y a_y n_y$, with $\{a_y\}_{y\in\Omega}$ in $\mathcal{R}$ vanishing outside some finitely generated ideal of Ω, in its standard topology.

Remarks. If Ω is finite (resp., finitely generated as coideal of itself) then $\rho_\pm\theta$ is nilpotent (resp., locally nilpotent) on $\mathcal{M}$.

5.3. **Proof of 1.3(a)–(c) and 1.4(a)–(d).** We first prove the claims about c'_y in 1.4(a) and (c). Let $c'\in\mathcal{M}$. If $c'=\theta(c')$ and $(1-\rho_-)(c')=t_y$ for some $y\in\Omega$, then $(1-\rho_-\theta)(c')=t_y$ so

$$c'=(1-\rho_-\theta)^{-1}t_y=\sum_{n\in\mathbb{N}}(\rho_-\theta)^n t_y\in t_y+\sum_{x<y}v^{-1}\mathbb{Z}[v^{-1}]t_x.$$

On the other hand, if $c':=\sum_{n\in\mathbb{N}}(\rho_-\theta)^n t_y$, then

$$c'=(1-\rho_-\theta)^{-1}t_y=(1-\rho_-\theta)^{-1}(1-\rho_-)t_y\in\ker(1-\theta)$$

and $(1-\rho_-)c'=t_y$ since $(1-\rho_-)(\rho_-\theta)^n=0$ for $n>0$.

Now set $c'_y:=(1-\rho_-\theta)^{-1}t_y=\sum_x p_{x,y}t_x$, so by above c'_y is as in 1.4(a). The characterization of c'_y shows that for fixed y, $\{p_{x,y}\}_{x\in\Omega}$ is the unique family of elements of $\mathcal{R}$, with $p_{x,y}=0$ for x outside some finitely generated ideal of Ω, such that $p_{y,y}\in 1+v^{-1}\mathbb{Z}[v^{-1}]$, $p_{x,y}\in v^{-1}\mathbb{Z}[v^{-1}]$, for $x\neq y$ and $p_{x,y}=\sum_z r_{x,z}\overline{p}_{z,y}$ for all x. This gives the existence and uniqueness of $p\in\mathcal{I}$ in 1.3(a) and shows that $c'_y=\sum_x p_{x,y}t_x$ for the $p\in\mathcal{I}$ so defined. The statement for q in 1.3(a) follows by applying that for p to the opposite poset Ω^{op} instead of Ω, and 1.3(b) is equivalent to 1.3(a). In 1.3(c), s is clearly unitriangular and $s=q\overline{p}=\overline{q}r\overline{p}=\overline{q}p=\overline{s}$.

The statements involving c_y instead of c'_y in 1.4(a)–(c) follow by symmetry from those for c'_y. The first assertion in 1.4(d) follows from the final paragraph of 5.2 and the final assertion of 1.4(d) follows from 1.3(b).

5.4. **Proof of 1.3(d), 1.4(e) and 1.7(b)–(c).** Assume henceforward that $\overline{r}=\hat{r}$. One has $p^{-1}=\overline{p}^{-1}\hat{r}$ by 1.3(b) and $\hat{q}=\hat{\overline{q}}\hat{r}$ from 1.3(a), so comparing gives the formula $p^{-1}=\hat{q}$. Then

$$s^{-1}=\overline{p}^{-1}q^{-1}=\hat{\overline{q}}\hat{p}=\hat{\overline{s}}.$$

The uniqueness in the last assertion of 1.3(d) follows from 1.3(a) applied with ϵ replaced by $-\epsilon$ and r by s.

The maps θ', $\rho'_\pm$ defined as in 1.4(e) are clearly continuous endomorphisms of $\mathcal{M}$ with θ' an involution and $\rho'_\pm$ orthogonal idempotents. The claim 1.7(b) follows immediately from the definitions. The condition $\hat{r}=\overline{r}$ implies by direct calculation that θ and θ' commute on $\mathcal{M}$. One has $\theta'(c'_y)=\sum_x\hat{\overline{p}}_{x,y}t_x=\sum_x(\overline{q}^{-1})_{x,y}t_x=c_y$, and hence $\theta'(\sum_y a_y c'_y)=\sum_x\hat{\overline{a}}_y c_y=\sum_x(\sum_y s_{x,y}\hat{\overline{a}}_y)c_x$. By the last paragraph of 5.2, one may identify $\mathcal{M}$ as the set of formal sums $\sum_x a_x c'_x$. Then applying 1.7(a) with ϵ replaced by $-\epsilon$ and r by s shows that $(\mathcal{M},\rho'_\pm,\theta')$ are in $\mathcal{A}''$, so 1.7(c) follows. Clearly, for fixed $y\in\Omega$, $t=t_y$ satisfies $t=\theta'(t)$ and $(1-\rho'_-)t=c'_y$. Uniqueness of the element $t\in\mathcal{M}$ satisfying these conditions, and the final formula in 1.4(e), follow on applying 1.4(a),(c) with ϵ replaced by $-\epsilon$ and r by s.

6. Some conjectures

In this section, we describe in more detail two of the many special instances of the standard formalism 1.3, relevant to Kazhdan-Lusztig polynomials and g-vectors of Eulerian posets, so as to be able to state some conjectures in these situations. For general background on Coxeter groups, Iwahori-Hecke algebras, Kazhdan-Lusztig polynomials etc see [17] and [11]. For g-vectors, see [23].

6.1. **Coxeter groups.** Let (W,S) be a Coxeter system with standard length function l and in 1.3 and 1.4, take $\Omega = W$ in Chevalley-Bruhat order; since Ω has a minimum element, $\mathcal{M}$ is a free $\mathbb{Z}[v,v^{-1}]$-module with basis $\{t_x\}_{x\in W}$, which we identify with the standard $\mathbb{Z}[v,v^{-1}]$-basis of the generic Iwahori-Hecke algebra $\mathcal{H}$ of (W,S) over $\mathbb{Z}[v,v^{-1}]$, with relations $t_xt_y = t_{xy}$ if $l(xy) = l(x)+l(y)$ and $t_r^2 = (v-v^{-1})t_r+t_1$ for $r \in S$ (so $\mathcal{M} = \mathcal{H}$). For x, y in W let $r_{x,y} = v^{l(x)-l(y)}R_{x,y}(v^2)$ where $R_{x,y}(q) \in \mathbb{Z}[q]$ is the Kazhdan-Lusztig R-polynomial, and take $\epsilon = 1$; then $\hat{r} = \overline{r}$ and θ is the Kazhdan-Lusztig involution

$$\sum_{w\in W} a_w(v)t_w \mapsto \sum_{w\in W} a_w(v^{-1})t_{w^{-1}}^{-1}$$

of $\mathcal{H}$. (Actually, θ and θ' are commuting ring involutions of $\mathcal{H}$ in this case.) One has $p_{x,y} = v^{l(x)-l(y)}P_{x,y}(v^2)$ (resp., $q_{x,y} = v^{l(x)-l(y)}Q_{x,y}(v^2)$) where $P_{x,y}(q) \in \mathbb{Z}[q]$ (resp., $Q_{x,y}(q) \in \mathbb{Z}[q]$ is the Kazhdan-Lusztig polynomial (resp., the inverse Kazhdan-Lusztig polynomial). It is conjectured (and known if (W,S) is crystallographic) that $P_{x,y}$ and $Q_{x,y}$ have non-negative coefficients.

Define elements $h_{x,y,z}, h'_{x,y,z} \in \mathbb{Z}[v,v^{-1}]$ by $c'_xc'_y = \sum_z h_{x,y,z}c'_z$ and $c'_xc_y = \sum_z h'_{x,y,z}c_z$. One has $h'_{x,1_W,y} = s_{x,y}$. It is known for crystallographic (W,S) (resp., finite Weyl groups (W,S)) that $h_{x,y,z}$ and $s_{x,y}$ (resp., $h'_{x,y,z}$) has non-negative coefficients. Let us say that an element $a \in \mathbb{Z}[v,v^{-1}]$ is evenly unimodal if it is (zero or) a finite sum of Gaussian integers $\frac{v^n-v^{-n}}{v-v^{-1}}$ for $n \in \mathbb{N}$.

Conjecture. For $x,y,z \in W$, $s_{x,y}$, $h_{x,y,z}$ and $h'_{x,y,z}$ are all evenly unimodal.

Remarks. The definition of $\mathcal{M}$ can be extended to "twisted" Chevalley-Bruhat orders on (W,S) [9] and the above conjectures extend to that situation. In that generality the conjectures for h and h' become instances of the same general conjecture, of which the generalized conjecture for s is not (in any obvious way as above) a special case.

6.2. Let Ω be a Eulerian poset [23], and take $\epsilon = 1$. Define $r_{x,y}$ by $r_{x,y} = (v-v^{-1})^{l(x,y)}$ where $l(x,y)$ is the length of a maximal chain from x to y if $x \leq y$, and $r_{x,y} = 0$ otherwise. Then the $p_{x,y}$ (resp., $q_{x,y}$) are (renormalization of) g-polynomials of a subinterval of Ω (resp., Ω^{op}). It is conjectured that if Ω is a (finite) Cohen-Macaulay lattice, then the $p_{x,y}$ (and hence the $q_{x,y}$ and the $s_{x,y}$) have non-negative coefficients, and if Ω is the face lattice of a polyhedral cone, this is known from [12] and [5] or [1].

Conjecture. If Ω is a finite Cohen-Macaulay (Eulerian) lattice, then $s_{x,y}$ is evenly unimodal for all $x,y \in \Omega$.

Remarks. It is known (see [26, 8.6]) that the f-vector of a convex polytope is not necessarily unimodal (even for simplicial polytopes); one may regard the above conjecture as providing an amended version of that (false) unimodality conjecture.

6.3. Interpretations and much stronger versions of the above conjectures can be given in terms of the representation categories described in [10]; combined with intersection cohomology techniques, these results imply that the $h_{x,y,z}$ and $s_{x,y}$ (resp., $h'_{x,y,z}$) are evenly unimodal for crystallographic Coxeter systems (resp., finite Weyl groups), as we shall show elsewhere. Conjecture 6.1 for dihedral Coxeter systems can be checked by direct computation, and we have checked it for type H_3 by computer. Conjecture 6.2 and the other cases of Conjectures 6.1 (and its generalization in the remark there) remain open.

6.4. Proposition 1.3 implies that in either 6.1 and 6.2, the family of polynomials $s_{x,y}$ determines the families of $p_{x,y}$ and of $q_{x,y}$. In 6.1, a recurrence formula can be given for $s_{x,y}$ in terms of the Kazhdan-Lusztig μ-function. It would be interesting to have in both 6.1 and 6.2 a more explicit formula for the $s_{x,y}$ in terms of data attached to the interval $[x, y]$, for instance similar to the generating function formula for $r_{x,y}$ in [9].

Remarks. In view of the formal symmetry between r and s in 1.3 under the assumption $\hat{r} = \overline{r}$, one might ask if analogues of the above conjectures on s hold for r. Namely, one might ask if, in 6.1 or 6.2, successive members of the sequence of integers arising as coefficients of $v^{l(x)-l(y)+2n}$ in $r_{x,y}$, for $n \in \mathbb{N}$, alternate in sign and the sequence of their absolute values is a unimodal sequence. This is trivially true in 6.2 and was for some time an open question in the situation 6.1, but has a negative answer in that situation as shown in [3].

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Department of Mathematics, Room 255 Hurley Building, University of Notre Dame, Notre Dame, Indiana, 46556-4618

E-mail address: `Dyer.1@nd.edu`