Review Questions for the Third Exam Solutions

1. Derivation of global error

Consider the differential equation:

$$\frac{dy}{dx} = f(x,y)$$

If we use the Backward Euler method approximation to this equation:

$$y_{k+1} = y_k + h\, f(\, x_{k+1}, y_{k+1}\,)$$

a. Derive the local error and the propagation error of this method and

b. What is the restriction on the step size h for stability?

This is in the notes.

2. Numerical solutions to non-linear differential equations:

Consider the differential equation and initial condition given by:

$$\frac{dy}{dx} = -\, y^2 \qquad y(1) = 1$$

a. Using the **trapezoidal rule** and a step size h = 1/2, solve this differential equation on the interval [1,2] (e.g., just three points including the initial value).

The trapezoidal rule is:

$$y_{k+1} = y_k + 0.5 * h * (\, f(t_k, y_k) + f(t_{k+1}, y_{k+1})\,)$$

Thus we have to solve the implicit equation at each step:

$$y_{k+1} = y_k + 0.5 * h * (\, -y_k^2 + -y_{k+1}^2\,)$$

which has the solution:

$$y_{k+1} = (\, (\, 1 + 2\, h\, y_k - h^2\, y_k^2\,)^{1/2} - 1\,)\ /\ h$$

or, with two iterations:

$y_0 = 1$ $y_1 = 0.646$ $y_2 = 0.483$

b. Recalculate the solution at x=2 using the same step size and the **Euler method**. Compare this result to that obtained in part a and to the exact solution y(x) = 1/x.

The Euler method is explicit, thus we have:

$$y_{k+1} = y_k + h * f(t_k,y_k)$$

or:

$$y_{k+1} = y_k + h * -y_k^2$$

which yields the two iterations:

$y_0 = 1$ $\quad$ $y_1 = 0.5$ $\quad$ $y_2 = 0.375$

Both methods are stable (e.g., the Euler method is within the bounds of numerical stability since $|1 + h J| < 1$) but the trapezoidal rule is much more accurate. The correct values are:

$y_0 = 1$ $\quad$ $y_1 = 2/3$ $\quad$ $y_2 = 1/2$

3. Statistics:

a. A series of measurements of the decay rate of a sample obtaining carbon-14 (the radioactive isotope of carbon used in radiocarbon dating) were taken as shown below. Using this information, calculate the mean, **population** variance (the variance for each measurement), and 95% confidence interval for the **average** decay rate assuming only random error. Show all of your work. The values given below are in counts per minute (cpm).

32.2
32.9
31.5
32.1
32.4
32.3
33.3
31.8
31.2
32.0

The sample mean and sample variance are unbiased estimates of the population mean and variances, thus:

$$\mu \approx \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \sigma^2 \approx s_{\bar{x}}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

which yield the numerical values:

$$\mu = 32.17 \qquad \sigma^2 = 0.382$$

Now the error in the average is just $1/N^{1/2}$ that of the population standard deviation. The 95% confidence interval is just the 2σ error bound. Thus the 95% confidence interval is:

95% confidence interval = [31.97 , 32.37]

b. Given that radioactive decay is governed by the Poisson distribution, in which the standard deviation of the number of decays observed in a given time is equal to the square root of the number of decays, estimate how long each of the above measurements took.

There are a number of ways to look at this. The simplest is via the precision argument: For measurements subject to Poisson statistics, the relative standard deviation is proportional to $1/(R^*t)^{1/2}$ where R * t is the total number of counts - R being the rate and t being the time. Thus:

$$\frac{\sigma}{\mu} = \frac{\sqrt{R * t}}{R * t}$$

Recognizing that R is just the mean and σ is as calculated above, we get:

$$t = \frac{\mu}{\sigma^2} \approx 84.2 \text{ minutes}$$

4. Regression analysis:

The matrix of covariance of an array of variables **x** is defined as:

$$\underset{\approx}{\Sigma_x^2} \equiv E\left\{\left(\underset{\sim}{x} - \underset{\sim}{\mu_x}\right)\left(\underset{\sim}{x} - \underset{\sim}{\mu_x}\right)^T\right\}$$

where **x** is a column vector. If we define the array **z** = **f** (**x**), derive an approximate formula for the matrix of covariance of **z**. Under what conditions will this approximation be valid?

This is in the notes.