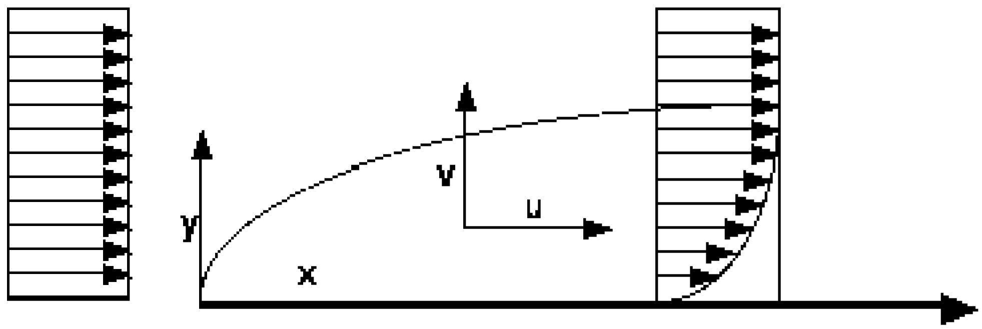
Problem 30 Due 4/27/01

A fundamental problem in fluid mechanics (and one you will see next fall in Transport I) is laminar flow past a flat plate at zero incidence. The geometry is depicted below:



The fluid flowing past the plate decelerates due to the no-slip condition on the surface of the plate (e.g., the velocity is zero at y = 0 because the plate isn't moving). The influence of the plate is primarily confined to a small region near the plate known at the boundary layer, which grows with distance down the plate. The equations governing this flow are the continuity equation (basically a statement of conservation of mass) and the boundary layer equation (a statement of conservation of momentum):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \boldsymbol{\nu} \frac{\partial^2 u}{\partial y^2}$$

where u is the velocity in the x-direction and v is the velocity in the y-direction. The quantity $\boldsymbol{\nu}$ is the kinematic viscosity, which has a value of about 0.01 cm^2/s for water and 0.1 cm^2/s for air. For two dimensional problems such as this one, it is convenient to introduce a streamfunction ψ, such that:

$$u = \frac{\partial \psi}{\partial y} \ , \quad v = - \frac{\partial \psi}{\partial x}$$

where ψ has been defined so that the continuity equation is automatically satisfied. Substituting the definition of the streamfunction into the boundary layer equation yields:

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = -\boldsymbol{\nu} \frac{\partial^3 \psi}{\partial y^3}$$

which is a nasty, non-linear mess. It turns out that this problem admits a similarity solution, in which the partial differential equation can be converted into an ordinary differential equation. The similarity rule and variable are:

$$\psi = (2\,U\,\nu\,x)^{\frac{1}{2}}\,f(\eta)\ ,\ \eta = \frac{y}{\left(\frac{2\,\nu\,x}{U}\right)^{\frac{1}{2}}}$$

such that:

$$u = U\,f'\,(\eta)$$

where U is the fluid velocity far from the plate. Plugging this back into the boundary layer equation yields the simple non-linear ODE:

$$f''' + f\,f'' = 0$$

with boundary conditions:

$$f(0) = f'(0) = 0\ ,\ f'(\infty) = 1$$

This equation is the famous Blasius equation you will study next fall.

Now we get to the problem I want you to solve: Determine the displacement thickness of the boundary layer defined by:

$$\delta^* = \frac{\delta}{\left(\frac{2\,\nu\,x}{U}\right)^{\frac{1}{2}}} = \int_0^\infty (1 - f')\,d\eta$$

and plot up the velocity profile in the boundary layer with this displacement thickness marked on the plot. The displacement thickness is the distance that fluid streamlines far from the plane are displaced by the boundary layer.

In order to solve this problem you will have to make some guesses about the missing initial condition f '' (0). You will have to adjust this value until the condition f ' (∞) = 1 is satisfied. You should play with the intial condition and observe the behavior of f ' at large η. It is useful to do this graphically until you get an idea about what is going on. Note that you don't really want to go to η = ∞, rather a value of 10 or 20 should suffice, as the deviation of f ' from unity should vanish exponentially in η. Once you have an idea of about where the correct initial condition should be, try using an automatic root finder such as fzero to nail it. Also, try tracking the integral of (1-f ') as an extra variable in the integration process to avoid having to do any quadrature later.