Cheg 258 Final Project **Due in my office at 6:00PM, 5/4/00**
Remember: You may not discuss this with others!

Viscous Resuspension in a Simple Shear Flow

During my dissertation work at Stanford, and subsequently with several of my students here at Notre Dame, we examined aspects of the problem of viscous resuspension. Resuspension is the process by which particles making up a settled bed (such as a stream bed or the sandy bottom of the ocean) are entrained in a fluid flowing over them. The resuspension process is complex, and has been studied for many years, but one aspect we have focused on is resuspension under viscous flow conditions, e.g., resuspension in the slow flow of a very viscous fluid. This parameter range also applies to the resuspension of very fine particles in less viscous fluids such as water.

Recently we have developed a new constitutive equation which describes the way in which particles migrate in a shear flow. In this problem we will apply this new constitutive equation to the resuspension process. Consider a simple shear flow such as that depicted below. In the laboratory, such a shear flow is produced by the tangential relative motion of two plates. At steady state the downward flux of the particles due to gravity is balanced by the upward flux due to the dispersive pressure produced between particles tumbling over one another in the shear flow. Thus the net particle flux is given by:

$$N_z = -\frac{2}{9} a^2 f \left[\dot{\gamma}_0 \, \mu_{eff} \frac{\partial \alpha}{\partial z} + \Delta\rho \; g \, \phi \right] \qquad 0 < \phi < \phi_m$$

where a is the particle radius, μ_0 is the viscosity of the suspending fluid, $\Delta\rho$ is the density difference between the particles and the fluid, $\dot{\gamma}_0$ is the apparent shear rate (the relative velocity of the two plates divided by the separation distance d), and g is the acceleration due to gravity. This equation is valid only when the volume fraction of the particles ϕ lies in the range $0 < \phi < \phi_m$, where ϕ_m is the maximum packing fraction of the particles, about 0.62. The model is characterized by three constitutive relations: the reduced normal stress in the velocity gradient direction α, the hindered settling factor f, and the suspension relative viscosity μ_r. These have been measured in independent experiments to be simple functions of concentration:

$$\alpha = 2.17\, \phi^3 \exp\left(2.34\, \phi\right) \qquad \mu_r = \frac{\exp\left(-2.34\, \phi\right)}{\left(1 - \dfrac{\phi}{\phi_m}\right)^3} \qquad f = \left(1 - \phi\right)^{5.1}$$

The problem is complicated by the deviation between the apparent relative viscosity of the suspension μ_{eff} and the measured relative viscosity of a suspension at the average concentration $\mu_r(\bar{\phi})$. If the density difference is very low, the dispersive pressure overwhelms the sedimentation due to gravity and the concentration is essentially uniform in the gap between the plates. Under these conditions the effective viscosity μ_{eff} is identical to $\mu_r(\bar{\phi})$. At higher sedimentation rates, however, the particles will settle and the concentration profile will become non-uniform. Under these conditions the effective viscosity is given by:

$$\mu_{eff} = \left[\frac{1}{d} \int_0^d \frac{1}{\mu_r} \, dz \right]^{-1}$$

and the conservation of particles condition is:

$$\bar{\phi} = \left[\frac{1}{d} \int_0^d \phi \, dz \right]$$

In this problem we are interested in the steady-state behavior as a function of sedimentation rate for some average concentration loaded into the gap between the plates. If we render the vertical coordinate z dimensionless with respect to the gap width d, then we have the dimensionless problem:

$$\frac{d\phi}{dz^*} = - S \frac{\phi}{\mu_{eff} \left[\dfrac{\partial \alpha}{\partial \phi} \right]} \qquad 0 < \phi < \phi_m \qquad S = \frac{\Delta\rho \; g \; d}{\dot{\gamma}_0 \; \mu_0}$$

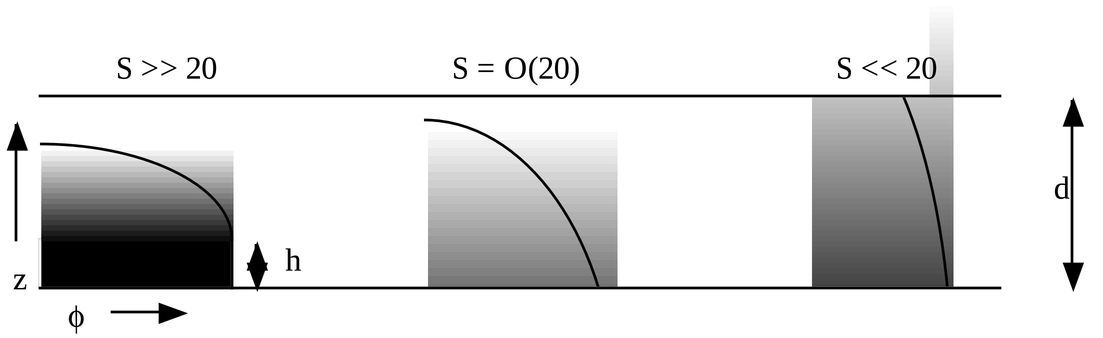
which is just a simple first order differential equation, but with a couple of integral conditions instead of an initial condition:

$$\mu_{eff} = \left[\int_0^1 \frac{1}{\mu_r} \, dz^* \right]^{-1} \qquad \bar{\phi} = \left[\int_0^1 \phi \, dz^* \right]$$

I want you to integrate this equation to determine (and plot) the concentration distribution for an average concentration $\bar{\phi} = 0.40$ and dimensionless sedimentation rate S = 10, 20, and 30. In addition, I want you to plot the effective viscosity μ_{eff} as a function of S over the range 1 < S < 50.

While the above sounds rather simple, the differential equation is very singular for larger values of S. Above a critical sedimentation rate the particles will settle to the point where there is a clear fluid layer, such as is depicted below. For still higher values of S there will also be a fully packed layer at the bottom plate of some dimensionless thickness h which will also have to be determined. Because of these singularities, and because the equations don't apply when the concentration is zero or at maximum packing (in both regions the concentration gradient is by definition zero), canned routines such as ode23 tend to fail. Thus while you can use canned software for small values of S, you will probably want to write your own code that can handle the specific singularity of this differential equation.

One point to consider in performing your integration is to treat the singularity where the concentration goes to zero -very- carefully. Any integration process will naturally predict a concentration beyond this point which is negative, clearly not a physical result. You can eliminate this problem by identifying the last point at which the concentration is above zero, and then determining the location of the top of the suspension layer analytically using the asymptotic form of the derivative as the concentration goes to zero.



What to do:

1). Write a routine which determines both the concentration distribution and the effective viscosity μ_{eff} as a function of S for the average concentration $\bar{\phi} = 0.40$. To start, just pick some value of μ_{eff} and the concentration at the bottom (think - for small values of S, what are reasonable guesses for these?), and integrate, playing with the values until everything matches (both integral conditions are satisfied) for a particular value of S (start with small values!).

2). Modify the code so that it can determine the correct values of μ_{eff} and the bottom concentration automatically for a particular value of S. The matlab routine fsolve is useful here, although sequential one dimensional root finding (do the bottom concentration first!) also works. The behavior of the integral is a very strong function of μ_{eff} and the bottom concentration, so good initial guesses are vital! Incrementing S and using the values of μ_{eff} and the bottom concentration for the last value of S as initial guesses for the next works pretty well!

3). Figure out what happens to the integration when S increases to the point where you get a clear fluid layer, and again to where you get a fully settled layer, and then modify the code to handle these pathological cases.

4). Finish the problem up by applying it to the desired range in S for the μ_{eff} plot, and the specific values of S for the concentration profile plots. Generate a table of μ_{eff} as a function of S for a reasonable range of values.

What to turn in:

1). Your program, which should be well commented and readable.

2). Your output, including a table of μ_{eff} for different values of S.

3). Two graphs, one showing the concentration profile for the three different values of S, and one plotting μ_{eff} as a function of S. On this second graph, mark the points at which a clear fluid layer forms, and at which a settled bed forms (e.g., determine the critical values of S and mark these points on the graph).

Final Project Evaluation Criteria:

The final project will be evaluated according to the following criteria:

25 points: Were you able to determine the correct value of μ_{eff} and the concentration profile for small values of S where the differential equation is non-singular?

25 points: Were you able to correctly determine μ_{eff} for the large values of S where the differential equation has singularities? Did you get the right values for the critical points (incipient clear fluid layer and incipient settled bed) in S?

20 points: Quality of your algorithms. Did you integrate the concentration, viscosity inverse, and cumulative density as a vector rather than sequentially? Did you deal with the singularities in a reasonably accurate manner? Did you determine the correct values of μ_{eff}, h, and the concentration at the bottom plate (for which h = 0) using an efficient root finding algorithm? Can the program be easily adapted to other values of the average concentration? This is not an exclusive list, just some of the ideas to consider in writing a high quality program.

20 points: Clarity of programming. Did you make effective use of functions, or did you try to cram everything into the main script? Did you use the global command in a restrained manner (remember that the global command is a meat axe - while it is extremely useful, it should also only be used when it significantly simplifies the program)? Is your program easy for us to follow? Is it well commented? Often you will have to pass your programs on to someone continuing your project in the future. If it is not clear and well commented, a program is useless to others and loses much of its value.

10 points: Clarity of presentation of graphical results (e.g., labelled axes & graphs, legend, title, etc.)

I hope that these criteria will make it easier for you to prepare the final versions of you program. Remember, do not discuss this with classmates until after 6:00PM on Thursday! You may talk to me or the TA's, however. Good Luck!