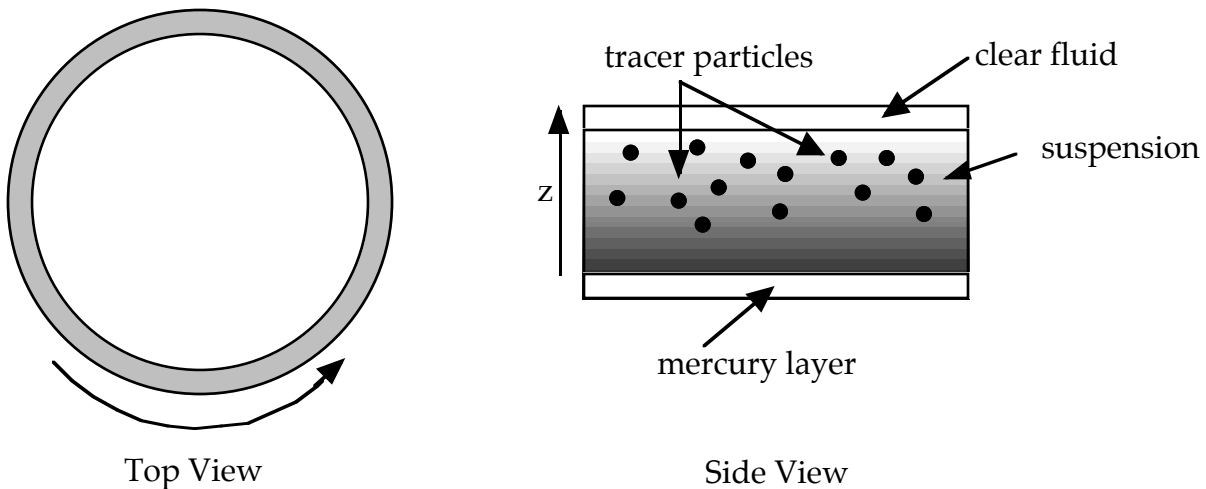


Remember: You may not discuss this with others!

**Bidisperse Resuspension in a Couette Viscometer:  
Steady-State Concentration Profiles**

In this problem we examine the steady-state concentration profile acquired by a suspension of particles which is both settling under the action of gravity, and resuspending under the action of shear-induced diffusion. The geometry is depicted below:



A suspension of negatively buoyant particles is placed in a Couette viscometer (the annular region between two concentric cylinders). A layer of mercury is used to seal the bottom of this annular region. The suspension is sheared by rotating the outer cylinder, producing a simple shear flow in the gap between the two cylinders. Because of gravity, the particles tend to settle out of the suspension and accumulate at the bottom. The applied shear, however, causes the particles to tumble over one another, leading to a diffusive flux which, at steady-state, balances sedimentation. The flux of particles due to these two mechanisms is given by:

$$N_s = - U_s f_s \phi_s - \hat{D}_\perp \dot{\gamma} a_s^2 \frac{\partial \phi_s}{\partial z}$$

In this equation  $\phi_s$  is the volume fraction (concentration) of the particles in the suspension,  $U_s$  is the Stokes sedimentation velocity (the settling velocity of a single particle in a pure fluid),  $f_s$  is the hindered settling factor which accounts for the presence of all of the other particles on the sedimentation velocity,  $\hat{D}_\perp$  is the dimensionless shear-induced effective diffusivity normal to the plane of shear (e.g., in the  $z$  direction in this geometry),  $\dot{\gamma}$  is the shear rate, and  $a_s$  is the radius of the particles. At steady-state the flux of particles is identically zero (the two mechanisms

balance), leading to a first order differential equation for the concentration as a function of position (z). The hindered settling factor and the effective diffusivity are both functions of concentration. These have been found empirically to be given by:

$$f_s = (1 - \phi_s)^{5.1}$$

and

$$\hat{D}_\perp = \frac{1}{3} \phi_s^2 \left( 1 + \frac{1}{2} \exp(8.8 \phi_s) \right)$$

In experiments in our laboratory, we are interested in the position of tracer particles which are larger than those making up the bulk of the suspension. We regard these tracers as dilute, so that we do not have to include tracer - tracer interactions in our analysis. Under these circumstances, it is possible to show that the tracer particle flux should be given by:

$$N_t = - U_s f_t \left( \frac{a_t}{a_s} \right)^2 \phi_t - \hat{D}_\perp^{\text{tr}} \dot{\gamma} a_s^2 \frac{\partial \phi_t}{\partial z} - \delta_\perp (\hat{D}_\perp - \hat{D}_\perp^{\text{sd}}) \dot{\gamma} a_s^2 \frac{\phi_t}{\phi_s} \frac{\partial \phi_s}{\partial z}$$

At steady-state this flux is also zero. In this equation  $f_t$  is the tracer particle hindered settling factor,  $\phi_t$  is the tracer concentration,  $a_t$  is the tracer particle radius,  $\hat{D}_\perp^{\text{tr}}$  is the tracer particle diffusivity (different from the effective diffusivity of a monodisperse suspension),  $\hat{D}_\perp^{\text{sd}}$  is the self-diffusivity for a monodisperse suspension (e.g.,  $\hat{D}_\perp^{\text{tr}}$  when  $a_t/a_s = 1$ ), and  $\delta_\perp$  is a function of the ratio  $a_t/a_s$  which describes the motion of the tracer particles due to a gradient in the suspending sphere concentration. It is this last quantity (in addition to confirmation of the whole constitutive equation) that we are trying to evaluate in our current (and future) experiments. The additional functions in the above equation are given by:

$$\hat{D}_\perp^{\text{tr}} \cong \frac{1}{2} \phi_s^2 \frac{a_s}{a_t}$$

$$\hat{D}_\perp^{\text{sd}} \cong \frac{1}{2} \phi_s^2$$

$$f_t \cong \frac{1 - \phi_s}{\mu_r} \quad \text{for } \frac{a_t}{a_s} \gg 1$$

where  $\mu_r$  is the relative viscosity (suspension viscosity divided by pure fluid viscosity) given by:

$$\mu_r = \left[ 1 + \frac{1.5 \phi_s}{1 - \phi_s/0.58} \right]^2$$

and, based on (very) preliminary results,

$$\delta_{\perp} \cong 1 + 4.75 \left( \frac{a_t}{a_s} - 1 \right)$$

It is convenient to render the above equations dimensionless. We choose as the characteristic length scale the height of the gap  $H$ . Thus letting  $z^* = z/H$  we get:

$$\frac{d\phi_s}{dz^*} = - \frac{U_s H}{\dot{\gamma} a_s^2} \frac{f_s \phi_s}{\widehat{D}_{\perp}}$$

and

$$\frac{d\phi_t}{dz^*} = - \frac{1}{\widehat{D}_{\perp}^{\text{tr}}} \left( \frac{U_s H}{\dot{\gamma} a_s^2} f_t \left( \frac{a_t}{a_s} \right)^2 \phi_t + \delta_{\perp} \left( \widehat{D}_{\perp} - \widehat{D}_{\perp}^{\text{sd}} \right) \frac{\phi_t}{\phi_s} \frac{d\phi_s}{dz^*} \right)$$

or, substituting the first equation into the second, we have the alternative expression:

$$\frac{d\phi_t}{dz^*} = - \phi_t \frac{U_s H}{\dot{\gamma} a_s^2} \frac{1}{\widehat{D}_{\perp}^{\text{tr}}} \left( f_t \left( \frac{a_t}{a_s} \right)^2 - f_s \delta_{\perp} \left( 1 - \frac{\widehat{D}_{\perp}^{\text{sd}}}{\widehat{D}_{\perp}} \right) \right)$$

as two dimensionless first order non-linear differential equations governing the suspension and tracer concentration profiles. The concentration profiles thus depend on the two dimensionless parameters:

$$\beta = \frac{a_t}{a_s} \quad \text{and} \quad \chi = \frac{U_s H}{\dot{\gamma} a_s^2}$$

We are interested in the concentration profiles as a function of  $\chi$  for the three tracer size ratios  $\beta = 1.5, 2, \text{ and } 4$ . I want the parameter  $\chi$  to be in the range [2 - 16], varying logarithmically (e.g., 2, 4, 8, 16). Do this for the average suspension concentration of  $\langle \phi_s \rangle = 0.40$  (this should be the average of  $\phi_s$  over the height of the Couette device), and normalize the tracer concentration to an average value of one. You should thus generate four graphs, one of the suspension concentration profile as a function of  $\chi$ , and one each of the tracer concentration profiles for each of the three size ratios. You will have multiple curves on each graph for the different values of  $\chi$  used. Be sure to use the legend and label commands to clearly label everything.

You will find that the above equations are fairly well behaved for small values of  $\chi$  (low sedimentation velocities). For larger values of  $\chi$ , however, the equation for the suspension concentration becomes singular at some height. This is

quite physical - there is a point where the concentration goes to zero, and there is simply a clear fluid layer devoid of particles above it. You will have to deal with this singularity in your calculations, thus you may not be able to use the canned routines ode23 or ode45 to do your computations (although you can try). Also, the above expressions are only defined for  $\phi_s, \phi_t > 0$  (negative concentrations are obviously not physical) thus you will probably need to "fix things" if whatever integrator you use is trying to evaluate these quantities for negative concentrations.

Remember, do not discuss this with classmates until after 5PM on Friday! You may talk to me or Jason, however. Good Luck!

## Final Project Evaluation Criteria

The final project will be evaluated according to the following criteria:

50 points: Did you get the correct answer, or how close did you get?

10 points: Clarity of presentation of graphical results (e.g., labelled axes & graphs, etc.)

The remaining 40 points will be determined based on the quality of the program you used to generate the answer. Particular issues are:

10 points: Appropriateness of integration algorithm. While the Euler method is useful for figuring out what is going on, a well-implemented higher order algorithm is necessary to receive full credit.

10 points: Dealing with the singularity at  $\chi = 8$  and 16 in a systematic and precise way. Acceptable methods include removal of the singularity analytically after truncation, or appropriate reduction of the step size before truncation. The key is to prevent relatively large errors from creeping into your answers.

10 points: Automatic determination of the bottom concentration. While manual shooting methods are fine for getting an idea of what the bottom concentration is as a function of  $\chi$ , some well-implemented automatic method is required for full credit.

10 points: Clarity of programming. Did you use the global command in a restrained manner (remember that the global command is a meat axe - while it is extremely useful, it should also only be used when it significantly simplifies the program)? Is your program easy for us to follow? Is it well commented? Often you will have to pass your programs on to someone continuing your project in the future. If it is not clear and well commented, a program is useless to others and loses much of its value.

I hope that these criteria will make it easier for you to prepare the final versions of you program.