

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9} \text{F/m}$,

Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7} \text{H/m}$.

Speed of light in free space: $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8 \text{ m/s}$.

Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r\mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Fields : $[\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_v]$

Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$

Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$

Ampere's Law : $[\oint_c \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S})]$, $[\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{d\epsilon_0 \mathbf{E}}{dt}]$

Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

Maxwell's equations in the phasor notation ($\mathbf{E} = \text{Re}[\hat{\mathbf{E}}e^{j\omega t}]$ & $\mathbf{B} = \text{Re}[\hat{\mathbf{B}}e^{j\omega t}]$) are given by:

$$\nabla \cdot (\epsilon_0 \hat{\mathbf{E}}) = \hat{\rho}_v,$$

$$\nabla \cdot \hat{\mathbf{B}} = 0,$$

$$\nabla \times \hat{\mathbf{E}} = -j\omega \hat{\mathbf{B}},$$

$$\nabla \times \frac{\hat{\mathbf{B}}}{\mu_0} = \hat{\mathbf{J}} + j\omega \epsilon_0 \hat{\mathbf{E}}.$$

In all of the above, $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ and $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J}_c = \sigma \mathbf{E}$, where σ is the conductivity of the material medium.

In addition to Maxwell's equations above in (B), the boundary conditions are -

For electric fields, $\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$, and $\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$.

For magnetic fields, $\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$ and $\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{\mathbf{E}} = \hat{E}_m e^{-\hat{\gamma}z}$, where

$$\hat{\gamma} = \alpha + j\beta,$$

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2}} [\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2},$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2}} [\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]^{1/2},$$

and the corresponding \hat{H} is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where

$$\hat{\eta} = \sqrt{\frac{\mu}{\epsilon - j\frac{\sigma}{\omega}}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1 + (\frac{\sigma}{\omega\epsilon})^2]^{1/4}} e^{j\frac{1}{2} \tan^{-1}(\frac{\sigma}{\omega\epsilon})}.$$

The skin depth of a conductive medium is $\delta = 1/\alpha$.

The total energy stored in the electric field is $W_E = \int_v \frac{1}{2} \epsilon |\mathbf{E}|^2 dv$ and in the magnetic field is

$W_M = \int_v \frac{1}{2} \mu |\mathbf{H}|^2 dv$, and therefore the total energy stored in a volume with both electric and magnetic fields is given by $W = W_E + W_M$.

Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$. The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P}_{av} = \frac{|E_m|^2}{2\eta_0} \mathbf{a}_z$ in vacuum or air, and by $\mathbf{P}_{av} = \frac{|E_m|^2}{2\hat{\eta}} e^{-2\alpha z} \cos \theta \mathbf{a}_z$, where $\theta = \frac{1}{2} \tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Static Electric and Magnetic Fields

- Under static conditions, the Electric field is a conservative field, and therefore can be defined as the gradient of a scalar electric potential, i.e.,

$$\mathbf{E} = -\nabla V$$

and equivalently, the potential difference between two points can be uniquely determined by the line integral of the electric field:

$$V_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}.$$

- The electric potential due to point, line, sheet, and volume charges are given by, respectively, $V_{point} = \frac{Q}{4\pi\epsilon r}$, $V_{line} = \int \frac{\rho_l dl}{4\pi\epsilon r}$, $V_{sheet} = \int \frac{\rho_s ds}{4\pi\epsilon r}$, $V_{vol} = \int \frac{\rho_v dv}{4\pi\epsilon r}$, where l, s, v stand for line, sheet, and volume respectively.

- The electric potential follows the principle of superposition; for example, for N point charges Q_i each located at r_i , the total potential is

$$V = \sum_{i=1}^N \frac{Q_i}{4\pi\epsilon r_i}.$$

- The capacitance of an object is a geometrical property that is defined as the ratio of the positive charge to the resulting potential difference between the conductors, i.e.,

$$C = \frac{Q}{V}.$$

- Gauss's law for electric field may now be re-cast in terms of the electric potential; this leads to Poisson's and Laplace's equations:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \text{ (Poisson's equation), \&}$$

$$\nabla^2 V = 0 \text{ (Laplace's equation, valid if } \rho_v = 0).$$

- A scalar field for the Magnetic field is not possible. However, the Magnetic field \mathbf{B} may be re-cast in terms of a Magnetic vector potential \mathbf{A} , such that

$$\mathbf{B} = \nabla \times \mathbf{A}, \text{ and}$$

$$\mathbf{A} = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') dv'}{R}, \text{ where } \mathbf{J}(\mathbf{r}') \text{ is the current density which produces the magnetic field.}$$

- Magnetic circuits may easily be solved by using the analogies with electrical circuits. The analogies are :

$$V \leftrightarrow NI \text{ (magnetomotive force),}$$

$$I \leftrightarrow \psi_m = B \cdot S, \text{ (} \psi_m \text{: magnetic flux, } B \text{: magnetic field, } S \text{: cross-sectional area),}$$

$$R \leftrightarrow R = \frac{L}{\mu S}, \text{ } L \text{: length of magnetic core, } \mu \text{: permeability,}$$

$$\text{and } \sigma \leftrightarrow \mu.$$

The analogies are valid as long as the permeability of the core is large enough to prevent substantial flux leakage out of the core.

- The ability of an object to produce magnetic flux in response to the current flowing through it is called the self inductance of the object, and is defined as

$$L_{11} = \frac{N_1 \psi_{11}}{I_1}.$$

Similarly, the mutual inductance between two conductors is given by

$$L_{12} = \frac{N_2 \psi_{12}}{I_1}.$$

E) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$)

$$d\mathbf{l} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z,$$

$$ds_x = dydz\mathbf{a}_x, ds_y = dx dz\mathbf{a}_y, ds_z = dx dy\mathbf{a}_z,$$

$$dv = dx dy dz.$$

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$)

$$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi d\mathbf{a}_\phi + dz\mathbf{a}_z,$$

$$ds_\rho = \rho d\phi dz\mathbf{a}_\rho, ds_\phi = d\rho dz\mathbf{a}_\phi, ds_z = \rho d\rho d\phi\mathbf{a}_z,$$

$$dv = \rho d\rho d\phi dz.$$

Spherical: (Unit Vectors: $\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$)

$$d\mathbf{l} = dr\mathbf{a}_r + r d\theta\mathbf{a}_\theta + r \sin\theta d\phi\mathbf{a}_\phi,$$

$$ds_r = r^2 \sin\theta d\theta d\phi\mathbf{a}_r, ds_\theta = r \sin\theta dr d\phi\mathbf{a}_\theta, ds_\phi = r dr d\theta\mathbf{a}_\phi,$$

$$dv = r^2 \sin\theta dr d\theta d\phi.$$

F) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a scalar field Φ ,

$$\text{grad } \Phi = \nabla\Phi = \frac{1}{h_1} \frac{\partial\Phi}{\partial u_1} \mathbf{a}_1 + \frac{1}{h_2} \frac{\partial\Phi}{\partial u_2} \mathbf{a}_2 + \frac{1}{h_3} \frac{\partial\Phi}{\partial u_3} \mathbf{a}_3, \text{ and}$$

$$\text{Laplacian } \Phi = \nabla^2\Phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial\Phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial\Phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial\Phi}{\partial u_3} \right) \right].$$

For a vector field $\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3$,

$$\text{div } \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(A_1 h_2 h_3)}{\partial u_1} + \frac{\partial(A_2 h_1 h_3)}{\partial u_2} + \frac{\partial(A_3 h_1 h_2)}{\partial u_3} \right], \text{ and}$$

$$\text{curl}(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a}_1}{h_2 h_3} & \frac{\mathbf{a}_2}{h_1 h_3} & \frac{\mathbf{a}_3}{h_1 h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, z	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors ($\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$)	$\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$	$\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ , 1	1, r , $r \sin\theta$