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# Fundamentals of Electromagnetic Fields and Waves: I

Fall 2006, EE 30348, Electrical Engineering, University of Notre Dame

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## 2nd Mid Term Exam: Solutions (11/09/2006)

Note: Please show your steps clearly and sketch figures wherever necessary. Points will be awarded for correct steps shown in the solutions.

### Fundamental Constants:

$$\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9} \text{F/m}, \mu_0 = 4\pi \times 10^{-7} \text{H/m}, c = \frac{1}{\sqrt{\epsilon_0\mu_0}} \approx 3 \times 10^8 \text{m/s}, \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega.$$

Note: There are Three problems in this exam, worth 20 Points. Answer all. All symbols have their usual meanings. Good luck!!

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### Problem 1 (4 Points): The Dielectric Stack

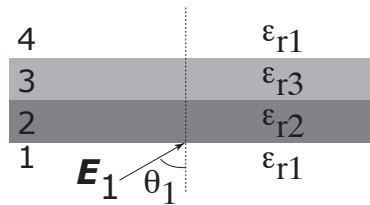


Figure 1: The dielectric stack (Problem 1)

Figure 1 above shows a stack of 2 dielectric layers, surrounded by air. The four regions are labeled 1-4, with 1 & 4 being air, and 2 & 3 are dielectrics. The relative dielectric constants of each layer are shown in the figure.

- a) If the electric field  $\mathbf{E}_1$  in air in region 1 below the stack points in the direction shown, find expressions for the field (both direction and magnitude) in the other three layers.

**Soln:** Electrostatic boundary conditions dictate that the tangential component of the electric field is continuous across the interfaces, and the normal component of the flux density (displacement) vector is continuous in the absence of surface charge. Therefore, writing  $E_{1n} = |\mathbf{E}_1| \cos \theta$  &  $E_{1t} = |\mathbf{E}_1| \sin \theta$ , we get directly that

$$E_{1t} = E_{4t} = E_{3t} = E_{2t}, \quad (1)$$

and since  $D_{in} = \epsilon_{ri} E_{in}$  for  $i = 1 - 4$ ,

$$\epsilon_{r1} E_{1n} = \epsilon_{r2} E_{2n} = \epsilon_{r3} E_{3n} = \epsilon_{r4} E_{4n}. \quad (2)$$

- b) Show that the field in region 3 is *independent* of the dielectric constant in region 2.

**Soln:** The resultant field in each layer ( $\mathbf{E}_i = E_{in}\mathbf{n} + E_{it}\mathbf{t}$ ) is dependent only on the *local* dielectric constant of that layer, and that in layer 1 where the field is known. Since  $\mathbf{E}_3 = (\epsilon_{r1}/\epsilon_{r3})E_{1n}\mathbf{n} + E_{1t}\mathbf{t}$  from equations 1 & 2, the field in layer 3 is independent of the dielectric constant of layer 2.

**Problem 2 (6 Points): Communication between Submarines and Aircrafts**

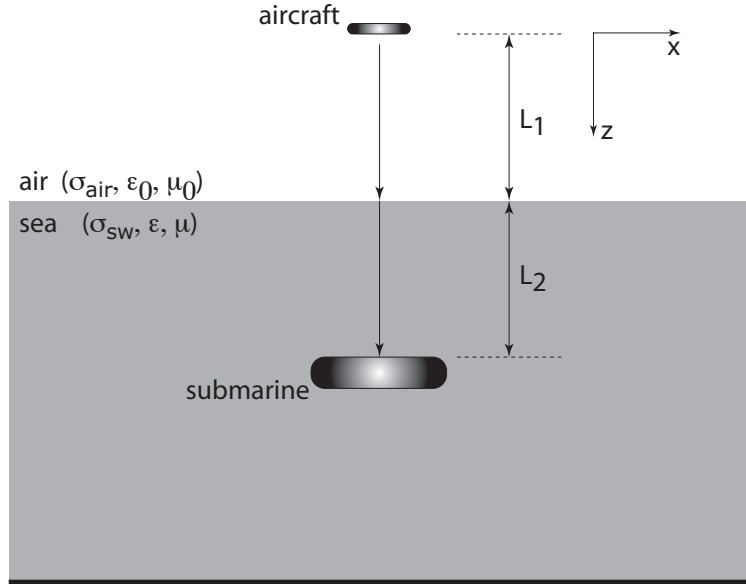


Figure 2: Deep-sea Communication (Problem 2)

Figure 2 above shows a submarine at a depth  $L_2$  from the surface of the sea. An aircraft at a height  $L_1$  above the surface of the sea wants to warn the submarine of an impending attack. The material properties of air and sea water are indicated in Figure 2. The aircraft uses an antenna that generates electromagnetic waves that travel towards the submarine. The electric field component of the wave generated at the aircraft antenna ( $z = 0$ ) is given by

$$\mathbf{E}(z = 0, t) = E_0 \cos(\omega t)\mathbf{a}_x \quad (3)$$

in V/m. Now the submarine also has a receiving antenna, but it can only detect electromagnetic waves whose electric field magnitude is *larger* than a small value given by  $E_1$  in V/m. Obviously  $E_0 \geq E_1$  for any communication to occur. Assume that the electromagnetic plane wave travels perpendicular to the surface of the sea as indicated by arrows, and neglect any reflection at the air/water interface.

- a) Assume first that the air is not conductive ( $\sigma_{air} = 0$  S/m). Find the depth  $L_2$ , such that if the submarine was any deeper, there is no communication possible. Your answer should be expressed in terms of all quantities defined in the problem.

**Soln:** The strength of the electric field decays only in a dissipative (i.e., conductive) medium. Hence, the EMag wave will propagate from the aircraft transmitter to the

air/water interface without any dissipation, implying that on the surface of water, the field magnitude is still  $E_0$ , same as at the source.

For propagation inside the sea, the field varies as

$$E_m^+(z > L_1) = E(z = L_1) \exp(-\alpha_{SW}[z - L_1]) \exp(-j\beta_{SW}[z - L_1]), \quad (4)$$

where  $E_{z=L_1} = E_0$ . Therefore, since the submarine depth should be at most  $L_2$  to detect the electric field magnitude  $E_1$ , the condition<sup>1</sup>

$$E_0 \exp(-\alpha_{SW}L_2) \geq E_1, \quad (5)$$

must be satisfied. This immediately yields  $L_2 \leq \ln(E_0/E_1)/\alpha_{SW}$  to be the maximum depth. Here,

$$\alpha_{SW} = \omega \sqrt{\mu_{SW}\epsilon_{SW}} \left( \frac{\sqrt{1 + \left(\frac{\sigma_{SW}}{\omega\epsilon_{SW}}\right)^2} - 1}{2} \right)^{1/2} \quad (6)$$

- b) Suppose it starts raining, and the air becomes conductive with conductivity  $\sigma_{air}$ . Find the new limit on the depth before they lose contact. Is it larger or smaller than your answer to part a)?

**Soln:** If the air becomes conductive, then there will be a decay of the electric field in both air and water, and the condition in equation (5) is modified to

$$E_0 \cdot \exp(-\alpha_{air}L_1) \cdot \exp(-\alpha_{SW}L_2) \geq E_1, \quad (7)$$

which immediately yields  $L_2 \leq \ln(E_0/E_1)/\alpha_{SW} - (\alpha_{air}/\alpha_{SW})L_1$  to be the new condition.  $\alpha_{air}$  is the same as expression (6), but with all subscripts  $SW \rightarrow air$ . Clearly, the submarine has to be closer to the surface of the sea to be able to detect the signal from the aircraft.

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<sup>1</sup>  $|\exp(-j\beta_{SW}[z - L_1])| = 1$  is used here.

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**Problem 3 (10 Points): Energy flow in a capacitor**

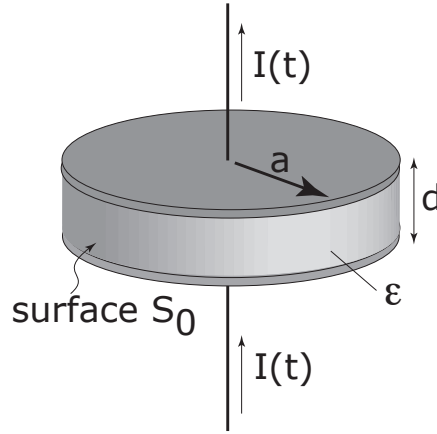


Figure 3: Power/Energy flow in a capacitor (Problem 3)

Figure 3 shows a parallel-plate capacitor (two infinitely conductive metallic discs) filled tightly with a cylindrical (radius  $a$  & height  $d$ ) dielectric material of dielectric constant  $\epsilon$ . The plates are large ( $a \gg d$ ), and therefore it is safe to neglect any fringing fields in this problem. The capacitor is connected to infinitely long wires that carry a time-dependent current given by

$$I(t) = I_0 \cos \omega t \quad (8)$$

in the direction shown. Answer the following questions:

- a) Find the electric field  $\mathbf{E}(t)$  vector inside the dielectric at any time  $t$ .  
 Hint: First find the sheet-charges  $\sigma_s(t)$  on the plates using the fact that the total charge on a metal plate at time  $t$  is given by  $Q(t) = \int_0^t I(t') dt'$ . Also, the total charge in the circuit is conserved. Then apply Gauss's law to get  $\mathbf{E}(t)$ .

**Soln:** To find the electric field, we first find the total charge on each plate at time  $t$  by using  $Q(t) = \int_0^t I(t') dt' = (I_0/\omega) \sin \omega t$ . The sheet charge on the plates is therefore  $\sigma_s(t) = \pm Q(t)/\pi a^2$ , and by Gauss's law, the resultant electric field is given by

$$\mathbf{E}(t) = \frac{\sigma_s(t)}{\epsilon} \mathbf{a}_z = \frac{I_0}{\pi a^2 \omega \epsilon} \sin \omega t \mathbf{a}_z \quad (9)$$

- b) Find the *total electric power* stored in the dielectric cylinder at any time  $t$ .

**Soln:** The total electric power stored is given by

$$\frac{\partial}{\partial t} W_E = \frac{\partial}{\partial t} \left[ \frac{1}{2} \int_{vol} \epsilon |\mathbf{E}(t)|^2 dv \right] = \frac{I_0^2 d}{\pi a^2 \epsilon \omega} \sin \omega t \cos \omega t, \quad (10)$$

where the result from equation 9 is used. Note that the same answer can be arrived at by noticing that the power is also given by  $I(t)V(t)$ , and  $V(t) = -\int \mathbf{E}(t) \cdot d\mathbf{l} = -E(t)d$ .

- c) Find the magnetic field intensity  $\mathbf{H}(t)$  at time  $t$  on the curved cylindrical surface  $S_0$  of the capacitor, as indicated in the figure.

Hint: Use Ampere's law.

**Soln:** Applying Ampere's law to a circular loop lying on the surface  $S_0$ , we get

$$\oint_c \mathbf{H}(t) \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_s \epsilon \mathbf{E}(t) \cdot d\mathbf{S}, \quad (11)$$

where the surface enclosed by the loop is the circular cross section of the dielectric of area  $\pi a^2$ . Since the *conduction* current is zero (the dielectric does not have any conductivity), the magnetic field is entirely due to the displacement current. Using the electric field  $\mathbf{E}(t)$  found in equation (9), we get that the magnetic field intensity is given by

$$\mathbf{H}(t) = \frac{I_0}{2\pi a} \sin \omega t \cdot \mathbf{a}_\phi. \quad (12)$$

We could have arrived at the same answer by deforming the surface such that it went through the wire carrying the current.

- d) Find the total power flowing across the entire surface  $S_0$  at time  $t$ . What is its relation to the total power stored in the capacitor you found in part a)?

**Soln:** The total power crossing the surface  $S_0$  is the surface integral of the Poynting vector,  $\mathbf{E}(t) \times \mathbf{H}(t)$ , which points in the  $-\mathbf{a}_\rho$  direction, into the surface. Using the time-dependent electric and magnetic field intensity derived in equations (9,12) above, we get the total power entering the surface to be

$$\int_{S_0} \mathbf{E}(t) \times \mathbf{H}(t) \cdot d\mathbf{S} = \frac{I_0^2 d}{\pi a^2 \epsilon \omega} \sin \omega t \cos \omega t, \quad (13)$$

which is exactly the same as in part (b).

- e) Explain why *all* the power flows from the sidewalls ( $S_0$ ), and none from the caps of the cylindrical dielectric.

**Soln:** Since the Poynting vector on the caps is along the  $-\mathbf{a}_\rho$  direction, whereas the surface normals on the caps point in the  $\pm \mathbf{a}_z$  direction,  $\mathbf{E}(t) \times \mathbf{H}(t) \cdot d\mathbf{S}$  is zero. In other words, no power crosses the caps, and all the power enters the dielectric through the sidewall surface  $S_0$ . This is possible since the charges in the wire carrying the current produce electric fields only inside the dielectric, and on its surface  $S_0$ . However, the conduction current in the wire and the displacement current in the dielectric produces a magnetic field *everywhere*. Thus, the Poynting vector is non-zero on the surface, and inside the dielectric.

End.

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