Fundamentals of Electromagnetic Fields and Waves: I

Fall 2006, EE 30348, Electrical Engineering, University of Notre Dame

1st Mid Term Exam (10/12/2006)
Note: Please show your steps clearly and sketch figures wherever necessary. Points will be awarded for correct steps shown in the solutions.

Fundamental Constants:

$\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 3 \times 10^8$m/s, $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega$.

Maxwell's Equations:

(Law): [Integral form], [Differential form]

(Gauss's Law for Electric Field): $[\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_v \rho_v dv]$, $[\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_v]$.
(Gauss's Law for Magnetic Field): $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$.
(Faraday's Law): $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_s \mathbf{B} \cdot d\mathbf{S}]$, $[\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}]$.
(Ampere's Law): $[\oint_c \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S}]$, $[\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}]$.

Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

All symbols have their usual meanings. Good luck!!

Problem 1

Answer the following short questions:

(a) (2 Points)
Show that Ampere's law directly leads to the charge continuity equation. What is the historic significance of this relationship between Ampere's law and charge continuity?

Solution: Ampere's law in the differential form is $\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}$. Taking the divergence of both sides, and making use of the identity $\nabla \cdot \nabla \times (...) = 0$ and Gauss's law for electric field $\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_v$, we get the continuity equation. Historically, the displacement current was introduced by Maxwell by noticing that if the $\frac{\partial \epsilon_0 \mathbf{E}}{\partial t}$ term is left out, charge continuity is violated.

(b) (2 Points)
The electric field due to an unknown volume charge distribution is given by $\mathbf{E} = 2x\mathbf{a_x} + 3bx\mathbf{a_y} + 2e^{-z}\mathbf{a_z}$ V/m, where b is a constant. Find the volume charge density at the point $(0, 0, 0)$.

Solution: Using Gauss's law for electric field, $\rho_v(x, y, z) = \nabla \cdot (\epsilon_0 \mathbf{E}) = \epsilon_0(2 - 2e^{-z})$ C/m^3. Therefore the charge density at $(0, 0, 0)$ is $\rho_v(0, 0, 0) = 0 \times \epsilon_0 = 0$ C/m^3.

(c) (2 Points)
Find the value of the constant b such that the electric field in part (b) is static.

Solution: For a static electric field, $\nabla \times \mathbf{E} = 3b\mathbf{a_z} = 0$. Therefore $b = 0$.

(d) (3 Points)
The electric field intensity of a <u>100 GHz</u> uniform electromagnetic plane wave propagating in source-free space is given in V/m by the phasor form

$$\hat{\mathbf{E}} = (2\mathbf{a_x} + 2\mathbf{a_y})e^{-j\beta_0 z}, \tag{1}$$

where β_0 is the phase factor (wavevector). i) Find β_0, ii) express the electric field in real-time form $\mathbf{E}(\mathbf{r}, t)$, and identify the iii) direction of propagation and the iv) polarization of the wave.

Solution:
i) Since $f = 10^{11}$Hz, $\omega = 2\pi \cdot 10^{11}$rad/s, and $\beta_0 = \omega/c = 2000\pi/3$ m^{-1}.
ii) $\mathbf{E}(\mathbf{r}, t) = Re[\hat{\mathbf{E}}e^{j\omega t}] = (2\mathbf{a_x} + 2\mathbf{a_y})\cos(2\pi \cdot 10^{11}t - 2000\pi z/3)$ V/m.
iii) The wave propagates along the +ve z direction.
iv) It is linearly polarized at 45° to the x-axis, since the $x-$ and y components are in phase, and $E_x = E_y$ at all times.

(e) (3 Points)
Find the magnetic field corresponding the the electric field in part (d). Express it first in phasor notation ($\hat{\mathbf{B}}$), and then in real-time form $\mathbf{B}(\mathbf{r}$,$t)$. Are the electric and magnetic fields of the EM wave perpendicular?

Solution:
Using Faraday's law in the phasor notation, $\hat{\mathbf{B}} = \frac{\nabla \times \hat{\mathbf{E}}}{-j\omega} = \frac{1}{c}[-2\mathbf{a_x} + 2\mathbf{a_y}]e^{-j\beta_0 z}$. Clearly, the real-time form is then given by $\mathbf{B}(\mathbf{r}, t) = Re[\hat{\mathbf{B}}e^{j\omega t}] = \frac{1}{c}[-2\mathbf{a_x} + 2\mathbf{a_y}]\cos(\omega t - \beta_0 z) = \mathbf{E}(\mathbf{r}, t)/c$ H/m. Since $\mathbf{E} \cdot \mathbf{B} = 0$, the electric and magnetic fields in the EM wave are perpendicular to each other, as they should be. Furthermore, $\mathbf{E} \times \mathbf{B}$ gives us the direction of wave propagation, the +ve z direction.

(f) (2 Points)
Find the value of the surface integral $\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$ on the closed surface shown in Figure 1(a). The charges inside and outside the sphere are shown in Coulombs. Is the electric field constant over the surface?

Solution:
From Gauss's law, the answer is -1 Coulomb. No, the electric field is changing on the surface of the sphere; it is not constant.

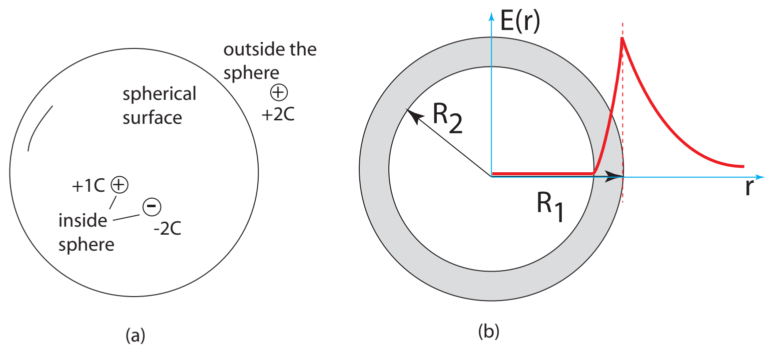

Figure 1: Problem 1(f) and Problem 2

Problem 2

(**6 Points**): A spherical shell of radii R_1 & R_2 is shown in Figure 1(b). The shaded region has a *uniform* charge density ρ_v C/m^3. Find the resulting electric field $\mathbf{E}$ at *all* points in space (i.e., for all r). Show all your steps and write down your arguments. Sketch the electric field magnitude vs r, the distance from the center of the sphere. (In case you need it, the differential radial surface element is $d\mathbf{S_r} = r^2 \sin\theta d\theta d\phi \mathbf{a_r}$, the radial spherical volume element is given by $dv = r^2 \sin\theta dr d\theta d\phi$, and $\int_0^\pi \sin\theta d\theta = 2$).

Solution: Applying Gauss's law in the integral form on a spherical surface with
a) $r < R_2$, we argue that the field has to be the same everywhere on the surface of this sphere, and since there is no charge enclosed, $\mathbf{E(r)} = 0$ for $r < R_2$.
b) For $R_2 < r < R_1$, we get $\epsilon_0 E(r) \cdot 4\pi r^2 = \rho_v \cdot \frac{4\pi}{3}(r^3 - R_2^3)$. Therefore, $\mathbf{E(r)} = \frac{\rho_v}{3\epsilon_0}(\frac{r^3 - R_2^3}{r^2})\mathbf{a_r}$.
c) Finally for $r > R_1$, $\mathbf{E(r)} = \frac{\rho_v}{3\epsilon_0}(\frac{R_1^3 - R_2^3}{r^2})\mathbf{a_r}$.
The maximum electric field is reached at the surface of the sphere, and there is a $\sim 1/r^2$ drop off outside the sphere. Inside the hollow cavity, the field is zero.