# Fundamentals of Electromagnetic Fields and Waves: I

Fall 2006, EE 30348, Electrical Engineering, University of Notre Dame

**Final Exam (12/12/2006)**
Note: Please show your steps clearly and sketch figures wherever necessary. Points will be awarded for correct steps shown in the solutions.

**Fundamental Constants**:

$\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 3 \times 10^8$m/s, $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega$.

**Note: There are Four problems in this exam, worth 30 Points. Answer all.**
All symbols have their usual meanings. Good luck!!

**Problem 1 (10 Points): Miscellaneous - need only short answers.**

a) Does the Poynting vector of an electromagnetic wave always point in the direction of wave propagation? If not, mention when it does not.

**Soln:** No, if the emag wave propagates in a conductive medium, the $\mathbf{E}$ and $\mathbf{B}$ fields go out of phase in a fraction of every time period. For this fraction of the period, the Poynting vector points opposite to the direction of propagation of the wave.

b) Why don't we have a scalar potential for magnetic fields?

**Soln:** A scalar potential exists for the electric field since the electric field due to a point charge is *conservative*. In other words, $\oint_c \mathbf{E} \cdot d\mathbf{l} = 0$ over any closed path, no matter what the charge distribution that leads to the electric field. This enables us to write the electric field as $\mathbf{E} = -\nabla\Phi$ where $\Phi$ is the scalar potential. Since $\oint_c \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_s \mathbf{J} \cdot d\mathbf{S}$, the $\mathbf{B}$ field does not enjoy the same privilege, and cannot be written as the gradient of a scalar, but as a curl of a vector ($\mathbf{B} = \nabla \times \mathbf{A}$).

c) Explain why for static conditions, the electric and magnetic fields are decoupled, whereas if they are time-varying, they are coupled.

**Soln:** Faraday's law $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ and Ampere's law $\nabla \times \mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$ imply that a time dependent magnetic field gives rise to an electric field, and vice versa. For time-independent fields, the electric field can be obtained from the charge distribution (from Gauss's law or Poisson's equation), and the magnetic field can be obtained from the current distribution using Faraday's law, *independent of each other*, implying that they are decoupled.

d) Find the total charge on a conducting sphere of radius $R$ which is at a potential $V$. The sphere is surrounded by air.

**Soln:** The capacitance of a spherical conductor of radius $R$ surrounded by air is $C = 4\pi\epsilon_0 R$ Farads. Therefore, if it is at a potential $V$, the total charge on it is $Q = CV = 4\pi\epsilon_0 RV$ Coulombs.

e) The region $0 < x < L$ of a large block of material with relative dielectric constant $\epsilon_r$ is filled with a volume charge density $\rho_v$. The charge is zero everywhere else. The electrostatic potentials at the boundaries are $\Phi(x = 0) = 0$ Volt, and $\Phi(x = L) = V_0$ Volt. Find the potential variation $\Phi(x)$ in $0 \leq x \leq L$.

**Soln:** The potential can be found by a direct solution of Poisson's equation:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} = -\frac{\rho_v}{\epsilon_r\epsilon_0}, \tag{1}$$

which yields the solution $\Phi(x) = -\frac{\rho_v x^2}{2\epsilon_r\epsilon_0} + Ax + B$. Using the two boundary conditions, $A$ & $B$ are easily determined, and the potential is found to be

$$\Phi(x) = \frac{\rho_v}{2\epsilon_r\epsilon_0}(xL - x^2) + \frac{V_0}{L}x. \tag{2}$$

It is evident that $\Phi(x)$ satisfies the two boundary conditions.

---

(contd...)

## Problem 2 (4 Points): A Magnetic Circuit

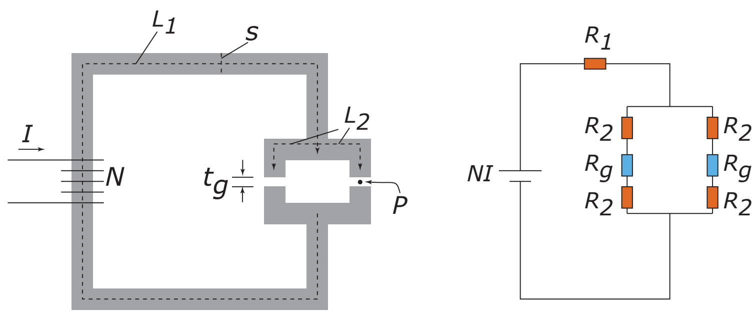


Figure 1: The magnetic circuit (Problem 2)

Figure 1 above shows a magnetic circuit. The ferromagnetic core has a relative permeability $\mu_r$. Find the magnetic field at point P. All relevant dimensions are shown in the figure, and the U-shaped regions are symmetric. Neglect fringing fields.

**Soln:** The reluctances of the equivalent magnetic circuit are $\Re_1 = L_1/\mu_r\mu_0 s$, $\Re_2 = L_2/\mu_r\mu_0 s$, and $\Re_g = t_g/\mu_0 s$. The total reluctance of the circuit is $\Re_{tot} = \Re_1 + (2\Re_2 + \Re_g)||(2\Re_2 + \Re_g) = \Re_1 + \Re_2 + \Re_g/2$. The magnetic flux in section '1' of the circuit is $\psi_m = NI/\Re_{tot}$. This flux divides equally between the two sections of the 'U' shape, and therefore the flux through each air gap is $\psi_m/2$. Therefore, the magnetic field at point $P$ is $B = \psi_m/2s = NI/2s\Re_{tot}$.

---

## Problem 3 (6 Points): Capacitance between a sphere & a conducting plane

Figure 2 shows a spherical conductor (radius $R$) at a distance $d$ from an infinitely large conducting plane. The conducting plane is grounded. Find the capacitance of the sphere-plate system. (Hint: Use the method of image charges.)

**Soln:** First, let's put a charge $Q$ on the conducting sphere. The capacitance between the sphere and the conducting plane is the given by $C = Q/\Delta V$, where $\Delta V$ is the potential difference between the sphere and the conducting plane. To find the potential difference, we make use of the method of image charges, and replace the conducting plane by a negative charge placed a distance $2d$ from the first, with a charge $-Q$. The potential at a distance $x$ from the 1st sphere along the straight line joining the two spheres is then given simply by the superposition of the two potentials:

$$V(x) = \frac{Q}{4\pi\epsilon_0 x} - \frac{Q}{4\pi\epsilon_0(2d - x)}. \tag{3}$$

The potential difference between the first sphere and the conducting plane is then given by

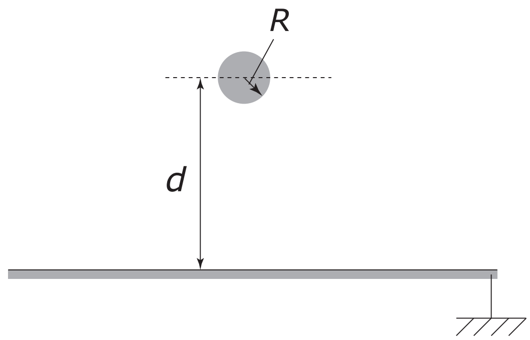


Figure 2: Capacitance between a sphere and a conducting plane (Problem 3)

$\Delta V = V(R) - V(d)$, given by

$$\Delta V = V(R) - V(d) = \frac{Q}{4\pi\epsilon_0}[(\frac{1}{R} - \frac{1}{2d-R}) - (\frac{1}{d} - \frac{1}{2d-d})] = \frac{Q}{4\pi\epsilon_0}(\frac{1}{R} - \frac{1}{2d-R}). \quad (4)$$

Therefore, the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{4\pi\epsilon_0}{\frac{1}{R} - \frac{1}{2d-R}}. \quad (5)$$

---

**Problem 4 (10 Points): Electric potential by solution of Laplace Equation**
Figure 3 shows two conducting plates (extending infinitely along the $z-$direction) separated by a small air gap (thickness $t$). Plate A extends from $-\infty \leq x \leq +\infty$, and plate B in $t \leq y \leq +\infty$. Plate A is grounded, and plate B is kept at a potential $V_0$. Answer the following questions:

a) Find the electric potential $\Phi(\rho, \phi, z)$ in region I.
Hint: Solve Laplace's equation in that region with the appropriate boundary conditions.

**Soln:** Laplace's equation in region I is $\partial^2\Phi(\phi)/\partial\phi^2 = 0$, with solutions of the form $\Phi(\phi) = A\phi + B$. The $\rho$ & $z$ derivatives are zero by symmetry. The two boundary conditions in region I ($\Phi(0) = 0$ & $\Phi(\pi/2) = V_0$) give us the values of $A$ & $B$; the solution is

$$\Phi_I(\rho, \phi, z) = \frac{2V_0}{\pi}\phi. \quad (6)$$

b) Find the potential in a similar fashion in regions II & III.
**Soln:** The potential in region II is

$$\Phi_{II}(\rho, \phi, z) = 2V_0(1 - \phi/\pi), \quad (7)$$

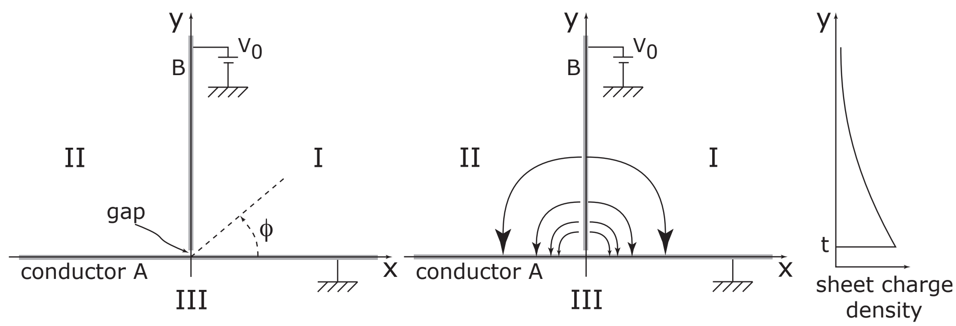


Figure 3: Electric potential and electric field (Problem 4)

and in region III, it is

$$\Phi_{III}(\rho, \phi, z) = 0. \tag{8}$$

c) Using parts a) & b), find the electric field $\mathbf{E}(\rho, \phi, z)$ in all three regions. Sketch the field lines.

**Soln:** The electric field in all regions is given by the negative gradient of the electric potential ($\mathbf{E} = -\nabla\Phi$). Hence, the electric field is given by

$$\mathbf{E}(\rho, \phi, z) = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\mathbf{a}_\phi = -\frac{2V_0}{\rho\pi}\mathbf{a}_\phi \tag{9}$$

in region I,

$$\mathbf{E}(\rho, \phi, z) = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\mathbf{a}_\phi = \frac{2V_0}{\rho\pi}\mathbf{a}_\phi \tag{10}$$

in region II, and

$$\mathbf{E}(\rho, \phi, z) = 0 \tag{11}$$

in region III. The field lines originate from plate B and terminate on plate A. They are circular, and the strength is the highest near the origin. There is no field in region III.

d) Find the sheet charge density on plate B as a function of $y$. Sketch the charge distribution.
**Soln:** The discontinuity in the electric field across plate B at $y$ is related to the sheet charge density at that point. The sheet density at $y$ is given by the relation

$$\frac{\rho_s(y)}{\epsilon_0} = [\mathbf{E}(\rho = y, \phi = \frac{\pi}{2} + \delta, z) - \mathbf{E}(\rho = y, \phi = \frac{\pi}{2} - \delta, z)] \cdot \mathbf{a}_\phi = \frac{4V_0}{\pi y}, \tag{12}$$

which directly yields the sheet charge as

$$\rho_s(y) = \frac{4\epsilon_0 V_0}{\pi y}. \tag{13}$$

Note that the charge on plate B is positive (field lines originate from there), and check that the dimensions are indeed C/m$^2$. The sheet charge distribution decreases as $\sim 1/y$, and is sketched in Figure 3.

e) If instead of being infinitely large along $y$, plate B extends from $t \leq y \leq L_y$, find the capacitance of the two-plate system per unit length along $z$. Neglect fringing fields (i.e., assume $L_y >> t$), and use your result from part d). Use the integral $\int dx/x = \ln x + C$ if necessary.

**Soln:** If plate B is of a finite length along $y$, then the total charge per unit $z-$length is given by

$$\rho_l = \int_t^{L_y} dy \rho_s(y) = \frac{4\epsilon_0 V_0}{\pi} \ln \frac{L_y}{t}, \tag{14}$$

and this immediately yields that the capacitance per unit length of the two-plate system is given by

$$C = \frac{\rho_l}{V_0} = \frac{4\epsilon_0}{\pi} \ln \frac{L_y}{t} \tag{15}$$

in units of F/m.

---

End.