
Fundamentals of Electromagnetic Fields and Waves: I

Fall 2008, EE 30348, Electrical Engineering, University of Notre Dame

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Mid Term Exam I - Solutions

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given at the **end** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1

Answer the following short questions.

a) Write down in your own words why for static electric field and magnetic fields $(\mathbf{E}, \mathbf{B})$, the two fields can be uncoupled (not related), whereas for time varying fields, they have to be related. (**2 Points**)

Answer: Static electric and magnetic fields imply that $\partial(...)/\partial t = 0$, which makes $\mathbf{E}$ & $\mathbf{B}$ independent of each other in Faraday's and Ampere's law. This implies that the electric and magnetic fields are uncoupled. For time varying fields, these two laws couple the two fields and therefore they are related to each other.

b) Stoke's theorem states that for a vector field $\mathbf{F}$, the relation $\oint_c \mathbf{F} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ holds, where s is the surface enclosed by the closed contour c. However, this relation holds only if $\mathbf{F}$ satisfies certain conditions. What are these conditions? (**1 point**)

Answer: The vector $\mathbf{F}$ and its spatial derivatives must be finite on the contour and on the surface area for Stoke's theorem to be applicable.

c) Show whether Stoke's theorem holds for the vector field $\mathbf{F} = \rho \mathbf{a}_\phi$ for a circle of radius ρ_0 in the $x-y$ plane centered at the origin. Necessary formulae are given at the end of this document. (**3 points**)

Answer: Since $\mathbf{F} = \rho \mathbf{a}_\phi$ and $\nabla \times \mathbf{F} = 2\mathbf{a}_\mathbf{z}$ is finite on the circle $\rho = \rho_0$ and in the surface enclosed by this circle, Stoke's theorem *must* hold. To prove it, we note that the contour integral is $\oint_c \mathbf{F} \cdot d\mathbf{l} = \oint_c (\rho_0 \mathbf{a}_\phi) \cdot (\rho_0 d\phi \mathbf{a}_\phi) = 2\pi\rho_0^2$. The surface integral of the curl is $\int_s (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_s (2\mathbf{a}_\mathbf{z}) \cdot (\rho d\rho d\phi \mathbf{a}_\mathbf{z}) = 2\int_0^{\rho_0} \rho d\rho \int_0^{2\pi} d\phi = 2\pi\rho_0^2$. The contour and the surface integrals are equal, which shows that Stoke's law holds.

d) An electromagnetic plane wave from an enemy aircraft is intercepted by a RADAR. The wavelength is measured to be $\lambda = 3$ mm, and the maximum electric field is measured to be $E_m = 1$ V/m. Find the frequency and the maximum magnetic field of the intercepted signal. (**2 Points**)

Answer: For an EM wave, $c = f\lambda$; therefore, $f = c/\lambda = 10^{11}$ Hz $= 100$ GHz. Also, since for a plane unidirectional EM wave, $|E| = c|B|$, the amplitude of the magnetic field is given by $|E|/c = 3.3 \times 10^{-9}$ T.

Problem 2 (**5 Points**)

An infinitely long cylindrical wire of radius ρ_0 aligned along the z-axis is filled with non-uniform positive charges. The volume charge density is given by $\rho_v(\rho, \phi, z) = K\rho$ in C/m^3, where ρ is the distance from the axis of the wire. Outside the wire is air.

Find the electric field for

a) $\rho > \rho_0$,

Answer: The electric field outside the cylinder is easily found by constructing a Gaussian cylinder with radius $\rho > \rho_0$ and height h. We apply Gauss's law for electric fields to this cylinder $\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_v \rho_v dv$. The RHS is given by $\int_{\rho=0}^{\rho_0} \int_{z=0}^{h} \int_{\phi=0}^{2\pi} (K\rho)(\rho d\rho d\phi dz) = 2\pi K h \rho_0^3/3$. The LHS is a surface integral - the two 'cap' surface vectors point along $\pm\mathbf{a_z}$, so they do not contribute to the integral since the electric field is radial ($\mathbf{E} = E(\rho)\mathbf{a}_\rho$) by symmetry. Thus, the LHS is $\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \int_{\phi=0}^{2\pi} \int_{z=0}^{h} (E(\rho)\mathbf{a}_\rho) \cdot (\rho d\phi dz \mathbf{a}_\rho) = \epsilon_0 E(\rho) \rho 2\pi h$. By equating the two sides, we obtain the electric field to be $\mathbf{E}(\rho > \rho_0) = (K\rho_0^3/3\epsilon_0\rho)\mathbf{a}_\rho$.

b) $\rho \leq \rho_0$

Answer: Following the same procedure as in part a), the Gaussian cylinder now has a radius $\rho < \rho_0$. Therefore, the RHS changes to $\int_{\rho=0}^{\rho} \int_{z=0}^{h} \int_{\phi=0}^{2\pi} (K\rho)(\rho d\rho d\phi dz) = 2\pi K h \rho^3/3$, i.e., less charge is now contained inside the Gaussian surface. The LHS remains the same as in a); equating, we find that the electric field is given by $\mathbf{E}(\rho \leq \rho_0) = (K\rho^2/3\epsilon_0)\mathbf{a}_\rho$.

c) Sketch the strength of the electric field as a function of the distance from the wire axis.

Answer: The electric field points radially outwards from the axis of the cylinder everywhere. Inside the cylinder, the field strength increases quadratically as ρ^2, and outside the cylinder, it drops as $1/\rho$. The field is continuous at the surface of the cylinder. Figure 1 shows a sketch of the strength of the electric field. The maximum electric field is reached at the surface of the cylinder - $E_{max} = K\rho_0^2/3\epsilon_0$.

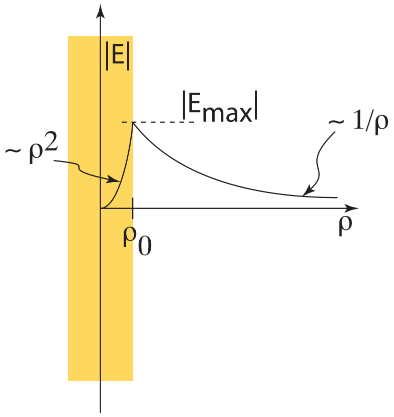

Figure 1: Electric field strength inside and outside the charged cylinder.

Problem 3 (**7 Points**)

The electric field of an EM plane wave is given by

$$\mathbf{E}(z,t) = [E_R \cos(\omega t - \beta_0 z) + E_L \cos(\omega t + \beta_0 z)]\mathbf{a_x}.$$

The units are in V/m.

Answer the following questions:

a) Explain qualitatively how the EM wave is propagating.

Answer: By noticing that the spatial components of the wave have both $+\beta_0 z$ and $-\beta_0 z$ parts, we realize that the EM wave is a superposition of two waves - one moving along the $+z$ direction, and the other moving along the $-z$ direction. The amplitudes of the two waves are related to the coefficients E_R and E_L respectively.

b) What is the polarization of the wave?

Answer: Since the electric field is always aligned along the x direction, the wave has linear polarization.

c) Write the electric field in the phasor notation, and identify the spatial component $\hat{\mathbf{E}}$.

Answer: Writing $[E_R \cos(\omega t - \beta_0 z) + E_L \cos(\omega t + \beta_0 z)]\mathbf{a_x} = Re[\hat{\mathbf{E}} e^{j\omega t}]$, we get $\hat{\mathbf{E}} = (E_R e^{-j\beta_0 z} + E_L e^{+j\beta_0 z})\mathbf{a_x}$.

d) Find the corresponding spatial component of the magnetic field $\hat{\mathbf{B}}$.

Answer: Using Faraday's law in the phasor notation, we note that $\hat{\mathbf{B}} = (-1/j\omega)\nabla \times \hat{\mathbf{E}}$. Thus, we find $\hat{\mathbf{B}} = (1/c)(E_R e^{-j\beta_0 z} - E_L e^{+j\beta_0 z})\mathbf{a_y}$.

e) Find the real magnetic field $\mathbf{B}(z,t)$.

Answer: The real magnetic field is given by $\mathbf{B}(z,t) = Re[\hat{\mathbf{B}} e^{j\omega t}] = (1/c)[E_R \cos(\omega t - \beta_0 z) - E_L \cos(\omega t + \beta_0 z)]\mathbf{a_y}$.

f) Show that $\mathbf{E} \perp \mathbf{B}$. Is $|\mathbf{E}|/|\mathbf{B}| = c$? Why?

Answer: Clearly, the electric and magnetic fields point in the x and y directions respectively. Therefore, $\mathbf{E} \cdot \mathbf{B} = (...)\mathbf{a_x} \cdot \mathbf{a_y} = 0$, and the electric and magnetic fields are perpendicular. However, the ratio of the magnitudes of the electric and magnetic field at any time and any position is

$$\frac{|\mathbf{E}(z,t)|}{|\mathbf{B}(z,t)|} = c \times \frac{[E_R \cos(\omega t - \beta_0 z) + E_L \cos(\omega t + \beta_0 z)]}{[E_R \cos(\omega t - \beta_0 z) - E_L \cos(\omega t + \beta_0 z)]},$$

which is not the speed of light c at every (z, t). This is because the wave is a superposition of a right-going wave of speed $+c$ and a left-going wave of speed $-c$. If the wave was purely right-going, $E_L = 0$, and we recover the ratio of $+c$, and if it was purely left-going, $E_R = 0$, and we would recover $-c$. For the superposed wave, the ratio clearly oscillates in space and time.

(List of Formulae in the next page...)

Formulae

A) Fundamental constants
Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 3 \times 10^8$ m/s.

B) Maxwell's Equations: [Integral Form], [Differential Form]
Gauss's Law for Electric Felds : $[\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S})]$, $[\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{d\epsilon_0 \mathbf{E}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

Maxwell's equations in the phasor notation ($\mathbf{E} = \text{Re}[\hat{\mathbf{E}} e^{j\omega t}]$ & $\mathbf{B} = \text{Re}[\hat{\mathbf{B}} e^{j\omega t}]$) are given by:

$\nabla \cdot (\epsilon_0 \hat{\mathbf{E}}) = \hat{\rho}_v$,

$\nabla \cdot \hat{\mathbf{B}} = 0$,

$\nabla \times \hat{\mathbf{E}} = -j\omega \hat{\mathbf{B}}$,

$\nabla \times \frac{\hat{\mathbf{B}}}{\mu_0} = \hat{\mathbf{J}} + j\omega\epsilon_0 \hat{\mathbf{E}}$.

C) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta dr d\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta dr d\theta d\phi$.

continued on the next page...

D) Formulae for div, grad, & curl expressions in various coordinate systems

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$

Use the metric coefficients (h_1, h_2, h_3), coordinates (u_1, u_2, u_3), and unit vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ from the Table above for the expressions of the grad, Laplacian, div, & curl in various coordinate systems.

For a scalar field Φ,
grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$, and
Laplacian $\Phi = \nabla^2\Phi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3})]$.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,

the divergence $(\nabla \cdot \mathbf{A})$ is given by

div $\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$,

and the curl $(\nabla \times \mathbf{A})$ is given by

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$