Fundamentals of Electromagnetic Fields and Waves: I

Fall 2008, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term II

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given in the **end** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1 (**6 Points**)

Answer the following short questions:

(a) Briefly ($\leq$ 3 sentences + a sketch) describe the microscopic picture behind the relative permittivity ϵ_r of a dielectric (1.5 Points).

Answer: The relative permittivity of a dielectric originates from the alignment of microscopic electric dipoles by an external electric field (Figure 1). The net polarization is $\mathbf{P} = \sum_i \mathbf{p}_i/\Delta v = \chi_e \epsilon_0 \mathbf{E}$. Therefore, the net electric flux density is $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E}$, where $\epsilon_r = 1 + \chi_e$.

(b) Briefly ($\leq$ 3 sentences + a sketch) describe the microscopic picture behind the relative permeability μ_r of a magnetic material (1.5 Points).

Answer: The relative permeability of a magnetic material originates from the alignment of microscopic current loops by an external magnetic field $\mathbf{H}$ (Figure 1). The net magnetization is $\mathbf{M} = \sum_i \mathbf{m}_i/\Delta v = \chi_m \mathbf{H}$. Therefore, the net magnetic field is $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H}$, where $\mu_r = 1 + \chi_m$.

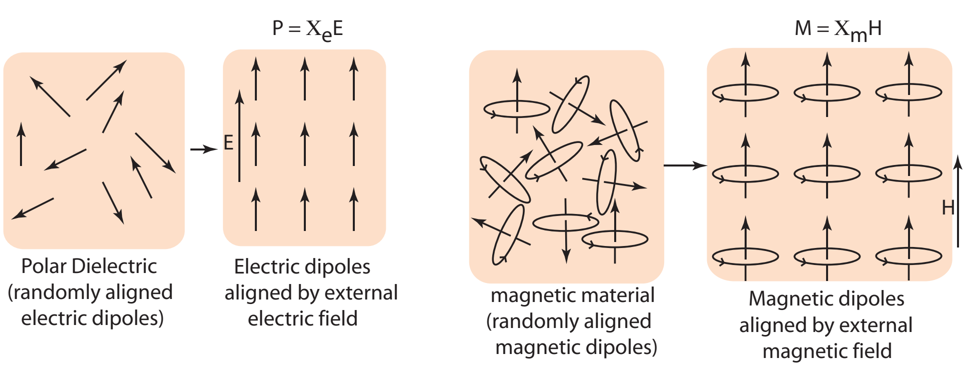

Figure 1: Sketches to accompany answers to problem 1 (a) & (b).

(c) An electromagnetic wave in free space enters a non-conductive dielectric medium of relative permittivity ϵ_r. Find the *change* in the speed (1 Point).

Answer: The speed in the dielectric is $v = c/\sqrt{\epsilon_r}$, therefore the *change* is $\Delta v = v - c = (\frac{1}{\sqrt{\epsilon_r}} - 1)c$.

(d) Show that the skin depth for an EMag wave of frequency f propagating in a medium (material parameters ϵ, μ) for a material of very low conductivity σ can be approximated by $\delta \approx \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$, and for a material of very high conductivity can be approximated by $\delta \approx 1/\sqrt{\pi f \mu \sigma}$. [You might need the Taylor expansion $(1+x)^n \approx 1 + nx$ for $x << 1$] (2 Points).

Answer: Since $\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1+(\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2}$, when the conductivity is very low, $\sigma/\omega\epsilon << 1$, $\sqrt{1+(\frac{\sigma}{\omega\epsilon})^2} \approx 1 + \frac{1}{2}(\frac{\sigma}{\omega\epsilon})^2$ and therefore $\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$, and the skin depth is $\delta = 1/\alpha \approx \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$. Similarly, if the conductivity is very high, $\sigma/\omega\epsilon >> 1$, and $[\sqrt{1+(\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2} \approx \sqrt{\frac{\sigma}{\omega\epsilon}}$, and hence $\alpha \approx \sqrt{\pi f \mu \sigma}$, and $\delta \approx 1/\sqrt{\pi f \mu \sigma}$. Here we used $\omega = 2\pi f$.

Problem 2 (**4 Points**)

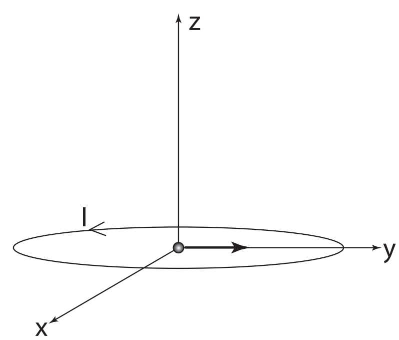

Figure 2: Problem 2

As shown in Figure 2, a positive charge q of mass m and velocity $v\mathbf{a_y}$ is located at the center of a circular wire loop of radius r in which a current I is flowing. Find the magnitude and direction of acceleration of the charge.

Answer: By Biot-Savart's law, the differential magnetic field at the center of the loop due to a loop line segment $d\mathbf{l} = rd\phi\mathbf{a}_\phi$ is given by $d\mathbf{B} = \frac{\mu_0 I(rd\phi\mathbf{a}_\phi)\times(-\mathbf{a}_\rho)}{4\pi r^2} = \frac{\mu_0 I d\phi}{4\pi r}\mathbf{a_z}$. Therefore the net magnetic field is got by integrating over the entire loop, $\mathbf{B} = \int_{\phi=0}^{2\pi} \frac{\mu_0 I d\phi}{4\pi r}\mathbf{a_z} = \frac{\mu_0 I}{2r}\mathbf{a_z}$. Due to this magnetic field, the Lorentz force on a point charge q with velocity $v\mathbf{a_y}$ is $\mathbf{F} = q(v\mathbf{a_y}) \times \mathbf{B} = \frac{q\mu_0 I v}{2r}\mathbf{a_x}$, and the acceleration is $\mathbf{a} = \mathbf{F}/m = \frac{q\mu_0 I v}{2mr}\mathbf{a_x}$, in the $+x$ direction.

Problem 3 (**10 Points**)

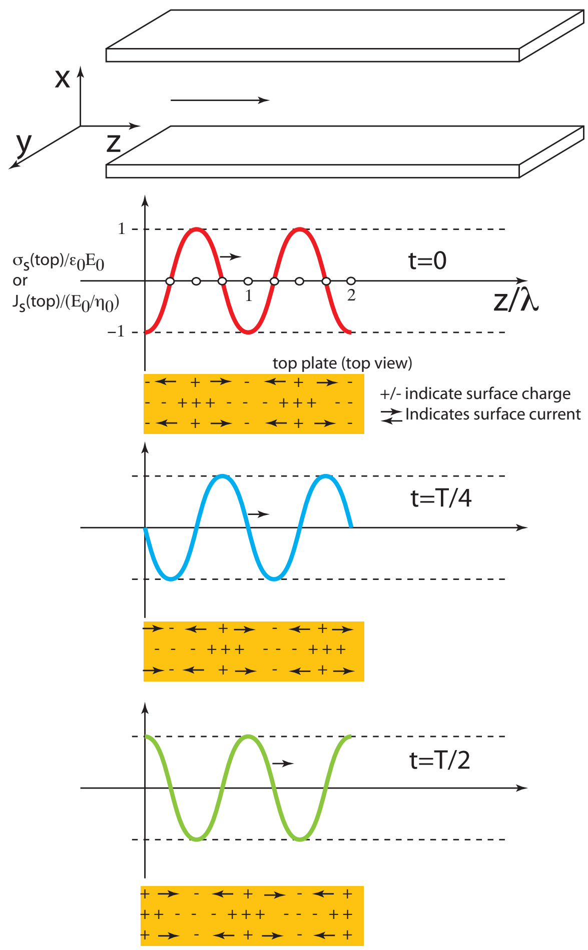

Figure 3: (Problem 3) An EMag plane wave propagating in a coplanar waveguide.

Refer to Figure 3. An electromagnetic plane wave of wavelength λ propagates along the $+z$ direction between two coplanar conductors. There is free-space between the conductors. Assume that the width of the waveguides are W and they extend infinitely in the $\pm z$ direction. Neglect fringing fields at all edges. The maximum electric field is E_0 and it points along the $\pm x$ axis.

(a) Write down expressions for the electric field $\mathbf{E}(z,t)$ and magnetic field intensity $\mathbf{H}(z,t)$ in the waveguide.

Answer: Let $\beta_0 = 2\pi/\lambda$, $\omega = 2\pi c/\lambda$, and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ be the wave impedance of free space. The electric field is $\mathbf{E}(z,t) = E_0 \cos(\omega t - \beta_0 z)\mathbf{a_x}$. The magnetic field intensity is $\mathbf{H}(z,t) = \frac{E_0}{\eta_0} \cos(\omega t - \beta_0 z)\mathbf{a_y}$.

(b) Using boundary conditions, find the surface charge densities that develop on the top and bottom plates, $\sigma_s^{top}(z,t)$ and $\sigma_s^{bottom}(z,t)$.

Answer: The electric field is normal to the conductor plates, therefore we need to consider the boundary condition for the normal component. We also know that $\mathbf{D}(z,t) = \epsilon_0 \mathbf{E}(z,t)$. For the top plate, $\mathbf{n} = -\mathbf{a_x}$. The boundary condition $\mathbf{n} \cdot \mathbf{D} = \sigma_s$ tells us that the surface charge density on the top plate must be $\sigma_s^{top}(z,t) = (-\mathbf{a_x}) \cdot (\epsilon_0 E_0 \cos(\omega t - \beta_0 z)\mathbf{a_x}) = -\epsilon_0 E_0 \cos(\omega t - \beta_0 z)$. Similarly, the surface charge density on the bottom plate is $\sigma_s^{bottom}(z,t) = +\epsilon_0 E_0 \cos(\omega t - \beta_0 z)$.

(c) Similarly, find the surface currents densities that flow on the top and bottom plates, $\mathbf{J}_s^{top}(z,t)$ and $\mathbf{J}_s^{bottom}(z,t)$.

Answer: The magnetic field is *parallel* to the conductor plates, therefore we need to consider the boundary condition for the tangential component, $\mathbf{n} \times \mathbf{H} = \mathbf{J}_s$. For the top plate, the surface current density is therefore given by $\mathbf{J}_s^{top}(z,t) = (-\mathbf{a_x}) \times (\frac{E_0}{\eta_0} \cos(\omega t - \beta_0 z)\mathbf{a_y}) = -\frac{E_0}{\eta_0} \cos(\omega t - \beta_0 z)\mathbf{a_z}$, and similarly, the surface current density on the lower plate is given by $\mathbf{J}_s^{bottom}(z,t) = (+\mathbf{a_x}) \times (\frac{E_0}{\eta_0} \cos(\omega t - \beta_0 z)\mathbf{a_y}) = +\frac{E_0}{\eta_0} \cos(\omega t - \beta_0 z)\mathbf{a_z}$.

(d) Sketch the surface charge density and the surface current densities for for the top plate in the region $0 \leq z \leq 2\lambda$ at time instants $t = 0, T/4$, and $T/2$, where $T = \lambda/c$ is the time period of the EM wave. Qualitatively explain the behavior.

Answer: The charge on the bottom plate is opposite in sign to that on the top plate, and the current flows in the opposite direction in the bottom plate. So let's sketch the charge and current in the top plate. For $t = 0$, the surface charge density on the top plate is given by $\sigma_s^{top}(z,t) = -\epsilon_0 E_0 \cos(\beta_0 z)$, and the surface current density is $\mathbf{J}_s^{top}(z,t) = -\frac{E_0}{\eta_0} \cos(\beta_0 z)\mathbf{a_z}$. Similarly, by substituting $t = T/4, T/2$, we obtain the charge and surface current densities as sketched in Figure 3. Clearly, in every cycle, there are positive and negative charge pileups, with the surface current being responsible for moving the charge density waves in an oscillatory fashion that is in phase with the EM wave that causes them.

(e) The above wave is called a transverse electromagnetic (TEM) mode of the waveguide, where both the electric and magnetic fields are transverse to the waveguiding direction. Show from boundary condition arguments that instead of the coplanar waveguide above, if it was a rectangular hollow conductor, it would not allow the propagation of a TEM mode.

Answer: If there were sidewalls, then the boundary conditions $E_{1t} = E_{2t}$ and $B_{1n} = B_{2n}$ cannot be satisfied, since both E_t and B_n are non-zero for the TEM mode for the sidewalls, but they are zero inside the metal sidewall. Therefore, a hollow rectangular waveguide cannot support a TEM mode of an EMag wave.

[End Questions]

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8$ m/s.
Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r\mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Felds : $[\oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot \mathbf{D} = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \mathbf{D} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

Maxwell's equations in the phasor notation ($\mathbf{E} = \text{Re}[\hat{\mathbf{E}}e^{j\omega t}]$ & $\mathbf{B} = \text{Re}[\hat{\mathbf{B}}e^{j\omega t}]$) are given by:

$\nabla \cdot \hat{\mathbf{D}} = \hat{\rho_v}$, $\nabla \cdot \hat{\mathbf{B}} = 0$, $\nabla \times \hat{\mathbf{E}} = -j\omega\hat{\mathbf{B}}$, and $\nabla \times \hat{\mathbf{H}} = \hat{\mathbf{J}} + j\omega\hat{\mathbf{D}}$.

In all of the above, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium. The Lorentz force on a charge q is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and the magnetic force on a line element of current is $d\mathbf{F} = Id\mathbf{l} \times B$. Coulomb's law is $\mathbf{F} = (q_1 q_2/4\pi\epsilon_0 r^2)\mathbf{a_{12}}$, and Biot-Savart's law is $d\mathbf{B} = \mu_0(Id\mathbf{l} \times \mathbf{a_R})/4\pi R^2$. All symbols have their usual meanings.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J_c} = \sigma\mathbf{E}$, where σ is the conductivity of the material medium.
In addition to Maxwell's equations above in (B), the boundary conditions are -
For electric fields, $\mathbf{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$, and $\mathbf{n} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$.
For magnetic fields, $\mathbf{n} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$ and $\mathbf{n} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$.
The bound polarization surface charge density at an interface is $\mathbf{n} \cdot (\mathbf{P_1} - \mathbf{P_2}) = -\rho_{ps}$ in C/m^2.
The bound magnetization surface current density at an interface is $\mathbf{n} \times (\mathbf{M_1} - \mathbf{M_2}) = \mathbf{J_{ms}}$ in A/m.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{E} = \hat{E_m}e^{-\hat{\gamma}z}$, where
$\hat{\gamma} = \alpha + j\beta$,
$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2}$,
$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]^{1/2}$,
and the corresponding $\hat{H}$ is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where
$\hat{\eta} = \sqrt{\frac{\mu}{(\epsilon - j\frac{\sigma}{\epsilon})}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1+(\frac{\sigma}{\omega\epsilon})^2]^{1/4}}e^{j\frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})}$.

The skin depth of a conductive medium is $\delta = 1/\alpha$.

The total energy stored in the electric field is $W_E = \int_v \frac{1}{2}\epsilon|\mathbf{E}|^2 dv$ and in the magnetic field is $W_M = \int_v \frac{1}{2}\mu|\mathbf{H}|^2 dv$, and therefore the total energy stored in a volume with both electric and magnetic fields is given by $W = W_E + W_M$.

Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.
The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\eta_0}\mathbf{a_z}$ in vacuum or air, and by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2|\hat{\eta}|}e^{-2\alpha z}\cos\theta\mathbf{a_z}$, where $\theta = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Static Electric and Magnetic Fields

• Under static conditions, the Electric field is a conservative field, and therefore can be defined as the gradient of a scalar electric potential, i.e.,
$\mathbf{E} = -\nabla V$
and equivalently, the potential difference between two points can be uniquely determined by the line integral of the electric field:
$V_{ab} = \int_a^b \mathbf{E}{\cdot}d\mathbf{l}$.

• The electric potential due to point, line, sheet, and volume charges are given by, respectively, $V_{point} = \frac{Q}{4\pi\epsilon r}$, $V_{line} = \int \frac{\rho_l dl}{4\pi\epsilon r}$, $V_{sheet} = \int \frac{\rho_s ds}{4\pi\epsilon r}$, $V_{vol} = \int \frac{\rho_v dv}{4\pi\epsilon r}$, where l, s, v stand for line, sheet, and volume respectively.

• The electric potential follows the principle of superposition; for example, for N point charges Q_i each located at r_i, the total potential is
$V = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon r_i}$.

• The capacitance of an object is a geometrical property that is defined as the ratio of the positive charge to the resulting potential difference between the conductors, i.e.,
$C = \frac{Q}{V}$.

• Gauss's law for electric field may now be re-cast in terms of the electric potential; this leads to Poisson's and Laplace's equations:
$\nabla^2 V = -\frac{\rho_v}{\epsilon}$ (Poisson's equation), &
$\nabla^2 V = 0$ (Laplace's equation, valid if $\rho_v = 0$).

• A scalar field for the Magnetic field is not possible. However, the Magnetic field $\mathbf{B}$ may be re-cast in terms of a Magnetic vector potential $\mathbf{A}$, such that
$\mathbf{B} = \nabla \times \mathbf{A}$, and
$\mathbf{A} = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')dv'}{R}$, where $\mathbf{J}(\mathbf{r'})$ is the current density which produces the magnetic field.

• Magnetic circuits may easily be solved by using the analogies with electrical circuits. The analogies are :
$V \leftrightarrow NI$ (magnetomotive force),
$I \leftrightarrow \psi_m = B \cdot S$, ($\psi_m$: magnetic flux, B: magnetic field, S: cross-sectional area),
$R \leftrightarrow R = \frac{L}{\mu S}$, L: length of magnetic core, μ: permeability,
and $\sigma \leftrightarrow \mu$.
The analogies are valid as long as the permeability of the core is large enough to prevent substantial flux leakage out of the core.

• The ability of an object to produce magnetic flux in response to the current flowing through

it is called the self inductance of the object, and is defined as
$L_{11} = \frac{N_1\psi_{11}}{I_1}$.
Similarly, the mutual inductance between two conductors is given by
$L_{12} = \frac{N_2\psi_{12}}{I_1}$.

E) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta drd\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta drd\theta d\phi$.

F) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a scalar field Φ,
grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$, and
Laplacian $\Phi = \nabla^2\Phi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3})$.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,
div $\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$, and

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$