Fundamentals of Electromagnetic Fields and Waves: I

Fall 2008, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term Exam

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given at the **end** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1

Answer the following short questions.

a) Write down in your own words why for static electric field and magnetic fields $(\mathbf{E}, \mathbf{B})$, the two fields can be uncoupled (not related), whereas for time varying fields, they have to be related. (**2 Points**)

b) Stoke's theorem states that for a vector field $\mathbf{F}$, the relation $\oint_c \mathbf{F} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ holds, where s is the surface enclosed by the closed contour c. However, this relation holds only if $\mathbf{F}$ satisfies certain conditions. What are these conditions? (**1 point**)

c) Show whether Stoke's theorem holds for the vector field $\mathbf{F} = \rho \mathbf{a}_\phi$ for a circle of radius ρ_0 in the $x-y$ plane centered at the origin. Necessary formulae are given at the end of this document. (**3 points**)

d) An electromagnetic plane wave from an enemy aircraft is intercepted by a RADAR. The wavelength is measured to be $\lambda = 3$ mm, and the maximum electric field is measured to be $E_m = 1$ V/m. Find the frequency and the maximum magnetic field of the intercepted signal. (**2 Points**)

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Problem 2 (**5 Points**)

An infinitely long cylindrical wire of radius ρ_0 aligned along the z-axis is filled with non-uniform positive charges. The volume charge density is given by $\rho_v(\rho, \phi, z) = K\rho$ in C/m^3, where ρ is the distance from the axis of the wire. Outside the wire is air.

Find the electric field for

a) $\rho > \rho_0$,
b) $\rho \leq \rho_0$, &
c) Sketch the strength of the electric field as a function of the distance from the wire axis.

Problem 3 (**7 Points**)

The electric field of an EM plane wave is given by

$$\mathbf{E}(z,t) = [E_R \cos(\omega t - \beta_0 z) + E_L \cos(\omega t + \beta_0 z)]\mathbf{a_x}.$$

The units are in V/m.

Answer the following questions:

a) Explain qualitatively how the EM wave is propagating.

b) What is the polarization of the wave?

c) Write the electric field in the phasor notation, and identify the spatial component $\hat{\mathbf{E}}$.

d) Find the corresponding spatial component of the magnetic field $\hat{\mathbf{B}}$.

e) Find the real magnetic field $\mathbf{B}(z,t)$.

f) Show that $\mathbf{E} \perp \mathbf{B}$. Is $|\mathbf{E}|/|\mathbf{B}| = c$? Why?

(List of Formulae in the next page...)

Formulae

A) Fundamental constants
Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 3 \times 10^8$ m/s.

B) Maxwell's Equations: [Integral Form], [Differential Form]
Gauss's Law for Electric Felds : $[\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S})]$, $[\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{d\epsilon_0 \mathbf{E}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

Maxwell's equations in the phasor notation ($\mathbf{E} = \text{Re}[\hat{\mathbf{E}} e^{j\omega t}]$ & $\mathbf{B} = \text{Re}[\hat{\mathbf{B}} e^{j\omega t}]$) are given by:

$\nabla \cdot (\epsilon_0 \hat{\mathbf{E}}) = \hat{\rho_v}$,

$\nabla \cdot \hat{\mathbf{B}} = 0$,

$\nabla \times \hat{\mathbf{E}} = -j\omega \hat{\mathbf{B}}$,

$\nabla \times \frac{\hat{\mathbf{B}}}{\mu_0} = \hat{\mathbf{J}} + j\omega\epsilon_0 \hat{\mathbf{E}}$.

C) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta dr d\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta dr d\theta d\phi$.

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D) Formulae for div, grad, & curl expressions in various coordinate systems

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$

Use the metric coefficients (h_1, h_2, h_3), coordinates (u_1, u_2, u_3), and unit vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ from the Table above for the expressions of the grad, Laplacian, div, & curl in various coordinate systems.

For a <u>scalar field</u> Φ,
grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$, and
Laplacian $\Phi = \nabla^2\Phi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3})]$.

For a <u>vector field</u> $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,

the divergence $(\nabla\cdot\mathbf{A})$ is given by

div $\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$,

and the curl $(\nabla\times\mathbf{A})$ is given by

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$