Fundamentals of Electromagnetic Fields and Waves: I

Fall 2008, EE 30348, Electrical Engineering, University of Notre Dame

<u>**Final Exam**</u>

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given in the <u>**end**</u> of this document.
- There are <u>**FOUR**</u> problems in this exam. Answer all.

Problem 1 (**8 Points**)

Answer the following short questions:

(a) What property of static electric fields allows us to define a scalar potential? (2 points)

(b) Why don't we have a magnetic scalar potential? (1 Point).

(c) In a tabular fashion, point out a few differences of EMag wave propagation in free space, non conductive dielectric media, and conductive media. (3 Points).

(d) Can the vector field $\mathbf{A} = 2x^2\mathbf{a_y}$ represent a magnetic vector potential? (2 Points).

Continued ...

Problem 2 (**6 Points**)

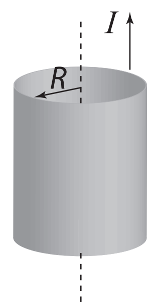

Figure 1: Problem 2

A current I flows in a long hollow thin-walled conducting cylinder of radius R (see Figure 1).

a) Qualitatively argue that

- there is a pressure (force per unit area) experienced by the walls,
- this pressure acts in a way to collapse the cylindrical conductor to its axis.

b) Now show quantitatively that the pressure on the walls is $p = \frac{\mu_0 I^2}{8\pi^2 R^2}$.

(*Hint*: Start by realizing that two wires carrying currents I_1 & I_2 in the same direction and separated by a distance r attract with a force per unit length $\frac{\mu_0 I_1 I_2}{2\pi r}$. Include the origin of this force in your explanation above.)

Problem 3 (**5 Points**)

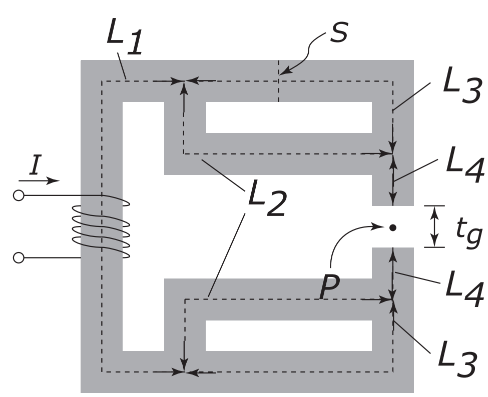

Figure 2: Problem 3

Refer to the magnetic circuit shown in Figure 2. The ferromagnetic core has a permeability μ, and the geometrical dimensions are shown (the cross-sectional area s is assumed to be uniform throughout the length of the core). You are given a current source which can deliver a current of I Amperes, and a long wire. Design the windings (i.e., how many turns N would you need) to create a magnetic field B_0 at point P in the air gap? Find the answer in terms of the variables given. Neglect any flux leakage from the magnetic core and fringing fields in the air gap.

Continued ...

Problem 4 (**11 Points**)

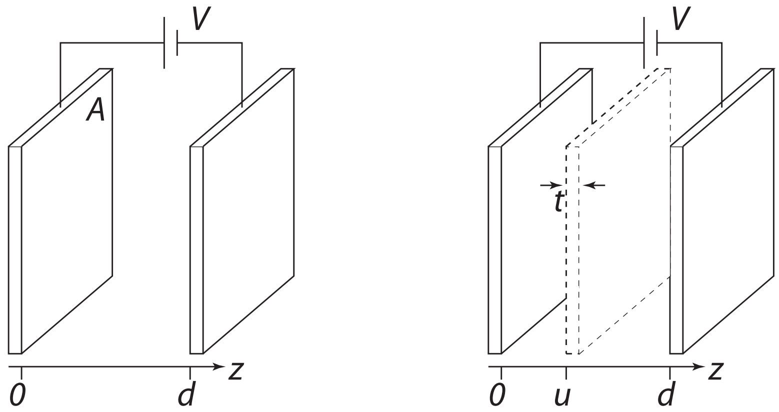

Figure 3: Problem 4

Consider the configurations of conducting plates shown in Figure 3. The left configuration shows a normal parallel-plate capacitor with free-space between the plates. The right configuration has a plate of thickness t inserted between the two plates, and this plate is electrically 'floating' (not connected to a battery). Neglect fringing fields. Answer the following questions -

a) Solve Laplace's equation between the plates to find the potential variation $\Phi(z)$ along z for the normal parallel plate capacitor. Find the resulting electric field vector $\mathbf{E}$. Use boundary conditions to find the net charge on the plates, and then show that the capacitance of the 'normal' parallel-plate capacitor is $C_0 = \epsilon_0 \frac{A}{d}$, where A is the area of the plates, and d the distance between them.

b) Find the net energy stored between the plates in the normal capacitor with the voltage V applied across them.

c) Sketch the magnitude of the electric field from $0 \leq z \leq d$ for the normal capacitor. Now consider the second configuration where the plate of thickness t is inserted. Sketch the magnitude of the electric field in this situation from $0 \leq z \leq d$.

d) From part c), give qualitative arguments why for the same voltage, the charges on the two wired plates is higher in the second configuration as compared to the normal configuration.

e) If the charge for the same voltage is higher, the second configuration must have a higher capacitance. Find this capacitance.

f) Find the work done in inserting the floating plate while the other plates are biased with the battery.

[End Questions]

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 1/\sqrt{\epsilon_0 \mu_0} \approx 3 \times 10^8$ m/s.
Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r \mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Felds : $[\oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot \mathbf{D} = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \mathbf{D} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

Maxwell's equations in the phasor notation ($\mathbf{E} = \text{Re}[\hat{\mathbf{E}} e^{j\omega t}]$ & $\mathbf{B} = \text{Re}[\hat{\mathbf{B}} e^{j\omega t}]$) are given by:

$\nabla \cdot \hat{\mathbf{D}} = \hat{\rho_v}$, $\nabla \cdot \hat{\mathbf{B}} = 0$, $\nabla \times \hat{\mathbf{E}} = -j\omega \hat{\mathbf{B}}$, and $\nabla \times \hat{\mathbf{H}} = \hat{\mathbf{J}} + j\omega \hat{\mathbf{D}}$.

In all of the above, $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ and $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium. The Lorentz force on a charge q is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and the magnetic force on a line element of current is $d\mathbf{F} = I d\mathbf{l} \times B$. Coulomb's law is $\mathbf{F} = (q_1 q_2 / 4\pi\epsilon_0 r^2)\mathbf{a_{12}}$, and Biot-Savart's law is $d\mathbf{B} = \mu_0 (I d\mathbf{l} \times \mathbf{a_R})/4\pi R^2$. All symbols have their usual meanings.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J_c} = \sigma \mathbf{E}$, where σ is the conductivity of the material medium.
In addition to Maxwell's equations above in (B), the boundary conditions are -
For electric fields, $\mathbf{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$, and $\mathbf{n} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$.
For magnetic fields, $\mathbf{n} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$ and $\mathbf{n} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$.
The bound polarization surface charge density at an interface is $\mathbf{n} \cdot (\mathbf{P_1} - \mathbf{P_2}) = -\rho_{ps}$ in C/m^2.
The bound magnetization surface current density at an interface is $\mathbf{n} \times (\mathbf{M_1} - \mathbf{M_2}) = \mathbf{J_{ms}}$ in A/m.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{E} = \hat{E_m} e^{-\hat{\gamma} z}$, where
$\hat{\gamma} = \alpha + j\beta$,
$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2}} [\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2}$,
$\beta = \omega \sqrt{\frac{\mu\epsilon}{2}} [\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]^{1/2}$,
and the corresponding $\hat{H}$ is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where
$\hat{\eta} = \sqrt{\frac{\mu}{(\epsilon - j\frac{\sigma}{\epsilon})}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1 + (\frac{\sigma}{\omega\epsilon})^2]^{1/4}} e^{j\frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})}$.

The skin depth of a conductive medium is $\delta = 1/\alpha$.

The total energy stored in the electric field is $W_E = \int_v \frac{1}{2}\epsilon|\mathbf{E}|^2 dv$ and in the magnetic field is $W_M = \int_v \frac{1}{2}\mu|\mathbf{H}|^2 dv$, and therefore the total energy stored in a volume with both electric and magnetic fields is given by $W = W_E + W_M$.

Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.
The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\eta_0}\mathbf{a_z}$ in vacuum or air, and by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2|\hat{\eta}|}e^{-2\alpha z}\cos\theta\mathbf{a_z}$, where $\theta = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Static Electric and Magnetic Fields

• Under static conditions, the Electric field is a conservative field, and therefore can be defined as the gradient of a scalar electric potential, i.e.,
$\mathbf{E} = -\nabla V$
and equivalently, the potential difference between two points can be uniquely determined by the line integral of the electric field:
$V_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$.

• The electric potential due to point, line, sheet, and volume charges are given by, respectively, $V_{point} = \frac{Q}{4\pi\epsilon r}$, $V_{line} = \int \frac{\rho_l dl}{4\pi\epsilon r}$, $V_{sheet} = \int \frac{\rho_s ds}{4\pi\epsilon r}$, $V_{vol} = \int \frac{\rho_v dv}{4\pi\epsilon r}$, where l, s, v stand for line, sheet, and volume respectively.

• The electric potential follows the principle of superposition; for example, for N point charges Q_i each located at r_i, the total potential is
$V = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon r_i}$.

• The capacitance of an object is a geometrical property that is defined as the ratio of the positive charge to the resulting potential difference between the conductors, i.e.,
$C = \frac{Q}{V}$.

• Gauss's law for electric field may now be re-cast in terms of the electric potential; this leads to Poisson's and Laplace's equations:
$\nabla^2 V = -\frac{\rho_v}{\epsilon}$ (Poisson's equation), &
$\nabla^2 V = 0$ (Laplace's equation, valid if $\rho_v = 0$).

• A scalar field for the Magnetic field is not possible. However, the Magnetic field $\mathbf{B}$ may be re-cast in terms of a Magnetic vector potential $\mathbf{A}$, such that
$\mathbf{B} = \nabla \times \mathbf{A}$, and
$\mathbf{A} = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')dv'}{R}$, where $\mathbf{J}(\mathbf{r'})$ is the current density which produces the magnetic field.

• Magnetic circuits may easily be solved by using the analogies with electrical circuits. The analogies are :
$V \leftrightarrow NI$ (magnetomotive force),
$I \leftrightarrow \psi_m = B \cdot S$, ($\psi_m$: magnetic flux, B: magnetic field, S: cross-sectional area),
$R \leftrightarrow R = \frac{L}{\mu S}$, L: length of magnetic core, μ: permeability,
and $\sigma \leftrightarrow \mu$.
The analogies are valid as long as the permeability of the core is large enough to prevent substantial flux leakage out of the core.

• The ability of an object to produce magnetic flux in response to the current flowing through

it is called the self inductance of the object, and is defined as
$L_{11} = \frac{N_1 \psi_{11}}{I_1}$.
Similarly, the mutual inductance between two conductors is given by
$L_{12} = \frac{N_2 \psi_{12}}{I_1}$.

E) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta dr d\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta dr d\theta d\phi$.

F) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a scalar field Φ,
grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$, and
Laplacian $\Phi = \nabla^2\Phi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3})$.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,
div $\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$, and

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$