Fundamentals of Electromagnetic Fields and Waves: I

Fall 2007, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term Exam

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given in the **last page** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1

Answer the following short questions.

a) Write down a few reasons why we are studying Electromagnetic Fields and Waves. (**2 Points**)

Answer: Can vary upon your perspective, but among other things it helps us understand and design every device/machine that involves the flow of currents and/or light.

b) The Divergence theorem states that for a vector field $\mathbf{F}$, the relation $\oint_s \mathbf{F} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{F} dv$ holds, where s is the closed surface enclosing the volume v. However, this relation holds only if $\mathbf{F}$ satisfies certain conditions. What are these conditions? Does the vector $\mathbf{F} = x\mathbf{a_x}$ qualify for application of the divergence theorem for a sphere centered at the origin? (**2 points**)

Answer: The Divergence theorem holds only if $\mathbf{F}$ is **finite** on the surface s and its divergence $\nabla \cdot \mathbf{F}$ is **finite** in the volume v. Since $\mathbf{F} = x\mathbf{a_x}$ is finite everywhere, and $\nabla \cdot \mathbf{F} = 1$ is also finite everywhere, this vector field does indeed qualify for the application of the Divergence theorem to *any* surface and the enclosed volume.

c) A FM radio station transmits electromagnetic plane waves at a frequency of $f = 100$ MHz. Find the wavelength of the signal. If the maximum electric field strength of the signal is 0.1 V/m, what is the maximum magnetic field for the signal? (**2 Points**)

Answer: The wavelength is $\lambda = c/f = 3$ meters. The magnetic field amplitude is $B_{max} = E_{max}/c = 3.33 \times 10^{-10}$ Tesla.

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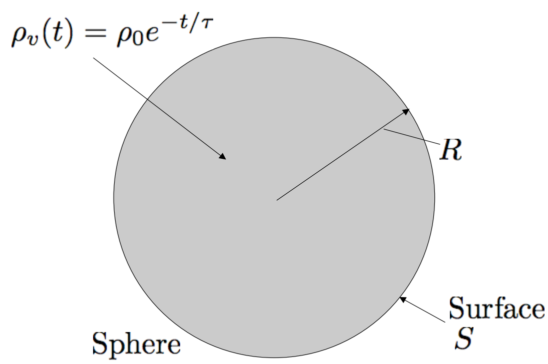

Figure 1: A spherical volume with decaying volume charge.

Problem 2 (**5 Points**)

A spherical region of radius R is shown in Figure 1. It has a volume charge distribution that is decaying with time according to the relation $\rho_v(t) = \rho_0 e^{-t/\tau}$. Here ρ_0 and τ are constants that characterize the decay process. The volume charge density is uniformly distributed inside the sphere at all times.

a) Use the charge continuity relation (given in the last page) to find the current density vector $\mathbf{J}(t)$ on the surface S of the sphere at time t. Explain each step in the process, and perform any consistency checks (dimensions, etc) for your answer.

Answer: By symmetry, the current density vector has to point radially outward, therefore assume $\mathbf{J} = J(r)\mathbf{a_r}$. The charge continuity equation states that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} = \frac{\rho_0}{\tau} e^{-t/\tau}.$$

To find the vector $\mathbf{J}$, integrate both sides over the volume of the sphere, and then apply the Divergence theorem:

$$\int_v \nabla \cdot \mathbf{J} dv = \int_v \frac{\rho_0}{\tau} e^{-t/\tau} dv = \oint_s \mathbf{J} \cdot d\mathbf{S}$$

Since the charge density is uniform inside the sphere, the middle term is just $\frac{\rho_0}{\tau} e^{-t/\tau} \times \frac{4}{3}\pi R^3$, and since the current density is uniform on the surface of the sphere, the surface integral on the right is just $J(R) \times 4\pi R^2$, which immediately yields

$$\mathbf{J}(r) = \frac{\rho_0 R}{3\tau} e^{-t/\tau} \mathbf{a_r}.$$

The current density is indeed in Amps/m^2.

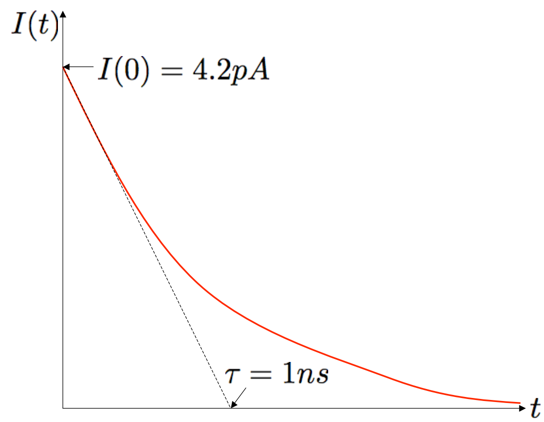

Figure 2: A spherical volume with decaying volume charge.

b) For $\rho_0 = 1.0$ C/m^3, $\tau = 1$ ns, and $R = 100$ nm, find the magnitude of the total *current* $I(t)$ flowing out of surface S as a function of time t. Sketch its value vs time.

Answer: The total current is given by $\oint_s \mathbf{J} \cdot d\mathbf{S}$, which evaluates to

$$I(t) = \frac{4\pi\rho_0 R^3}{3\tau} e^{-t/\tau} = 4.2 \times 10^{-12} e^{-t/1ns}$$

in Amperes, a really small current. The sketch in Figure 2 shows the value of the current with time.

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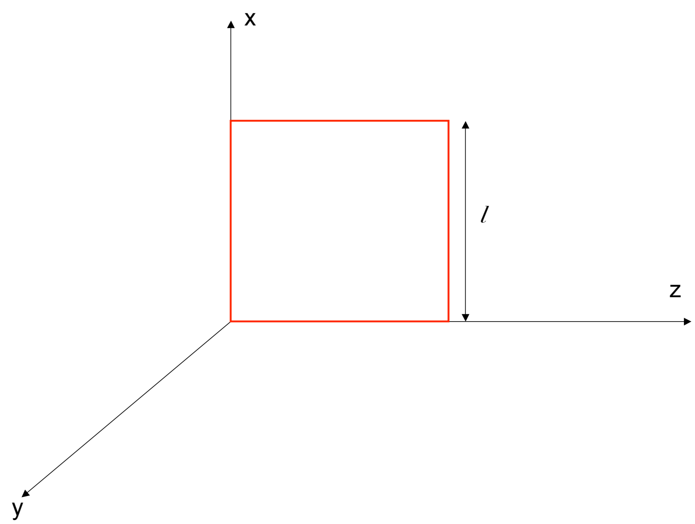

Figure 3: A square loop interacts with an electromagnetic wave.

Problem 3 (**9 Points**)

The electric field vector of an electromagnetic plane wave propagating in free space is given by

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\omega t - \beta_0 z)\mathbf{a_x}$$

a) Identify the wavelength, frequency, polarization, and the direction of propagation of the plane wave.

Answer: The wavelength is $\lambda = 2\pi/\beta_0$, the frequency is $f = \omega/2\pi$, the wave is plane (or linearly) polarized, and is moving along the +ve z direction.

b) Write down the expression for the corresponding magnetic field $\mathbf{B}(\mathbf{r}, t)$.

Answer: The magnetic field is $\perp$ to both the electric field and the direction of wave propagation. Since $\mathbf{E} \times \mathbf{B}$ gives the direction of wave propagation and the ratio of the electric field and the magnetic field is the speed of light c, the magnetic field is given by

$$\mathbf{B}(\mathbf{r}, t) = \frac{E_0}{c} \cos(\omega t - \beta_0 z)\mathbf{a_y}.$$

c) Refer to Figure 3. A square conductive loop of side l and total resistance R_0 is now placed in the path of the electromagnetic wave as shown. Explain **qualitatively** why the electromagnetic wave will cause a current to flow in the loop.

Answer: Due to the plane wave, there is a time-varying magnetic field pointing along the y-direction. Since the magnetic field flux passing through the loop is changing with time, it will generate an emf by Faraday's law - $emf = -\frac{d\int_s \mathbf{B} \cdot d\mathbf{S}}{dt} = \oint_c \mathbf{E} \cdot d\mathbf{l}$. This emf will drive an electric current around the square loop.

d) Find the magnitude of the induced current flowing in the loop.
(You may or may not need the following relations:
$\int \sin(az)dz = -\frac{\cos(az)}{a}$+C,
$\int \cos(az)dz = \frac{\sin(az)}{a}$+C,
$\sin(a) - \sin(b) = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})$,
$\cos(a) - \cos(b) = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$)

Answer: The current is given by $I(t) = emf(t)/R_0$. The emf may be calculated in two ways - either by using the rate of change of the magnetic flux crossing the loop $-\frac{d\int_s \mathbf{B}\cdot d\mathbf{S}}{dt}$, or by evaluating the line integral of the electric field around the loop edge $\oint_c \mathbf{E} \cdot d\mathbf{l}$. Both ways will lead to the same answer.

The line integral of the electric field is somewhat easier. Since the electric field points in the $\mathbf{a_x}$ direction, the contributions to the line integral from the top and bottom edges of the loop is zero since the lines are perpendicular to the electric field. From the left and right edges, we get simply that the emf should be $[|\mathbf{E}(z=0,t)| - |\mathbf{E}(z=l,t)|] \times l$, or

$$emf(t) = E_0 l[\cos(\omega t) - \cos(\omega t - \beta_0 l)] = -2E_0 l \sin(\frac{\beta_0 l}{2})\sin(\omega t - \frac{\beta_0 l}{2}).$$

Check the dimensions: they are in volts. Therefore, the current flowing through the loop is given by $I(t) = emf(t)/R_0$, and it is clear that it oscillates with time with the same frequency as the plane wave that causes it. The loop can therefore be used to detect the frequency of the plane wave and can serve as an antenna. Note that the amplitude of the emf is $-2E_0 l \sin(\frac{\beta_0 l}{2})$, and it depends on the actual size of the loop as well as the wavelength of the plane wave. Question: For what size of the square is the signal maximized?

e) Assume that the length of the loop l is much smaller than the wavelength of the wave (show why this is equivalent to the condition $\beta_0 l \ll 1$). Simplify your answer to part d) using the approximation $\sin(x) \approx x$ for $x \ll 1$. Explain the significance of this simplification.

Answer: If $l << \lambda$, the relation $\beta_0 l = 2\pi l/\lambda << 1$ holds. Using the simplification $sin(\beta_0 l/2) \approx \beta_0 l/2$, the emf becomes

$$emf(t) = -2E_0 l \sin(\frac{\beta_0 l}{2})\sin(\omega t - \frac{\beta_0 l}{2}) \approx E_m \beta_0 l^2 \sin(\omega t - \frac{\beta_0 l}{2}).$$

Now $E_0\beta_0 l^2 = \omega B_0 l^2$, which translates to the total magnetic flux crossing the loop assuming the magnetic field is constant in z. Or in other words, when the loop is much smaller than the wavelength of the plane wave, the signal strength is proportional to the area of the loop l^2.