Fundamentals of Electromagnetic Fields and Waves: I

Fall 2007, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term II: Solutions

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given in the **end** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1 (4 Points)

Answer the following short questions (1 point each):

(a) Which three parameters describe the behavior of electric and magnetic fields and the propagation of electromagnetic waves inside material medium?

Solution: The three parameters are the electrical conductivity (σ), the relative dielectric constant (ϵ_r), and the relative magnetic permeability (μ_r).

(b) Of the three material parameters in part (a), only one leads to energy (or power) loss when an EM wave propagates. Which one? Why?

Solution: The electrical conductivity σ is the only parameter that contributes to energy (or power) loss. Energy can be stored and retrieved from the parts that go into the electric and magnetic dipoles (via ϵ_r and μ_r), but not from the conduction current I^2R loss since this energy is converted to mechanical vibrations or heat, which is not an electromagnetic wave.

(c) In order to gather information about outer space, why is it necessary to send out satellites and space telescopes outside the earth's atmosphere?

Solution: Since the earth's atmosphere contains various gases with varying degrees of conductivity (for example the ionosphere is very conductive), weak signals in the form of EM waves from outer space get attenuated strongly by the time they reach the earth's surface. To prevent this from happening, satellites and space telescopes are sent out of the atmosphere where they can collect much stronger signals and beam them back to the earth.

(d) What is the only property of an EM wave that does not change when it crosses from one medium to another? Find the change in the wavelength when an EM wave that had $\lambda = 500$ nm in air enters water with $\epsilon_r \approx 81$ and $\mu_r = 1$.

Solution: The frequency $f = \omega/2\pi$ of an EM wave remains constant when it crosses from one material medium to another. For an EM wave in any medium, $f = v/\lambda$ is constant, where v is the speed of the wave and λ is the wavelength. Also, in a material medium, $v = c/\sqrt{\epsilon_r \mu_r}$, where $c = 3 \times 10^8$ m/s is the speed of light in vacuum. Therefore the wavelength in water is

$$\lambda_{water} = (\frac{v_{water}}{c})\lambda_{air} = \frac{1}{\sqrt{81}}500 = 500/9 \sim 56nm. \tag{1}$$

Problem 2 (**6 Points**)

An infinitely long straight wire is carrying a current given by $I(t) = I_0 \cos \omega t$. A small conducting loop of area $\mathbf{S} = S\hat{\mathbf{n}}$ and resistance R_0 placed at a very large distance ρ_0 from the wire. Find

(a) The current flowing through this loop as a function of the angle θ the normal to the surface $\hat{\mathbf{n}}$ makes with the axis of the infinite wire, and

Solution: The magnetic field due to the infinite wire at the location of the small loop is given by

$$\mathbf{B} = \frac{\mu_0 I_0 \cos \omega t}{2\pi\rho_0}\mathbf{a}_\phi \tag{2}$$

If the surface vector $\hat{\mathbf{n}}$ of the small loop makes an angle θ with the axis of the wire which is along $\mathbf{a_z}$, then $\mathbf{a}_\phi \cdot \hat{\mathbf{n}} = \sin\theta$. In that case, the total magnetic flux passing through the small loop is given by

$$\phi_m = \int_s \mathbf{B} \cdot d\mathbf{S} \approx |\mathbf{B}| S \cos\theta = \frac{\mu_0 I_0 S \cos \omega t}{2\pi\rho_0} \sin\theta, \tag{3}$$

and the emf generated in the loop is then given by

$$V(t) = -\frac{d\phi_m}{dt} = \frac{\omega\mu_0 I_0 S \sin \omega t}{2\pi\rho_0} \sin\theta, \tag{4}$$

and since the resistance of the loop is R_0, the current that flows through it is

$$I(t) = \frac{V(t)}{R_0} = \frac{\omega\mu_0 I_0 S \sin \omega t}{2\pi\rho_0 R_0} \sin\theta. \tag{5}$$

We can check that the units do work out to Amps.

(b) The power (in Watts) picked up by the loop as a function of the angle. Can we connect the small loop to light up a bulb (in principle)? At what angle θ will the bulb glow the brightest if it does?

Solution: The power generated in the loop at any time t is given by $I(t)V(t) = I(t)^2 R_0 = V(t)^2/R_0$ in Watts. Clearly, this is given by

$$power(t) = \frac{[\frac{\omega\mu_0 I_0 S \sin \omega t}{2\pi\rho_0} \sin\theta]^2}{R_0}. \tag{6}$$

This is clearly ac power, and just like ordinary light bulbs that run on 50-60 Hz ac power, sure, it will light up. The loop has to be aligned in a way to maximize the magnetic flux crossing it for obtaining the highest power. This is achieved when the $\theta = \pi/2$, or the surface is perpendicular to the magnetic flux lines.

Problem 3 (**10 Points**)

An electromagnetic plane wave moves from air (ϵ_{r1}, μ_{r1}) into a wafer of silicon (ϵ_{r2}, μ_{r2}) at normal incidence (Poynting vector along the interface normal), as shown in Figure 1. $\mathbf{E_i}$, $\mathbf{H_i}$, and $\mathbf{P_i}$ as shown in the figure stand for the electric field, magnetic field intensity, and the Poynting vector of the *incident* EM wave respectively. The incident electric and magnetic field vectors are parallel to the Silicon-air interface, as shown. Just like for any wave incident on a material medium, a part of it will be transmitted and part of it will be reflected. As shown in the figure, the subscript "t" stands for the transmitted components, and "r" stands for the reflected components. Answer the following questions -

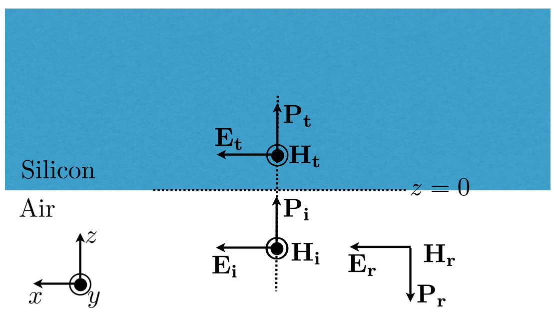

Figure 1: An electromagnetic plane wave moving from air into a silicon wafer.

(a) Call the wave impedance of air as η_1. Find the wave impedance η_2 in Silicon. (**0.5 Point**)

Solution: $\eta_1 = \eta_0\sqrt{\frac{\mu_{r1}}{\epsilon_{r1}}}$, and $\eta_2 = \eta_0\sqrt{\frac{\mu_{r2}}{\epsilon_{r2}}}$, where $\eta_0 \approx 377\Omega$ is the wave impedance of vacuum.

(b) The Poynting vector indicates the direction of wave propagation, as well as the movement of energy. Clearly, the reflected wave carries part of the energy back towards $-z$ (shown in Fig 1). Using this fact, and assuming that the reflected electric field points in the direction indicated, find the direction of the magnetic field of the reflected wave. (**0.5 Point**)

Solution: $\mathbf{P_r}$ is in the $-z$ direction, and $\mathbf{E_r}$ is along $+x$. Since $\mathbf{P_r} = \mathbf{E_r} \times \mathbf{H_r}$, $\mathbf{H_r}$ has to point in the $-y$ direction by the right hand rule.

(c) Now that all directions of the vectors are fixed, lets find the relationships between the fields in air and in Silicon. We can write the incident electric field in the phasor notation as $\hat{\mathbf{E_i}} = E_{mi}e^{-j\beta z}\mathbf{a_x}$ where E_{mi} is the amplitude of the incident electric field. Assume that there are no charges or currents on the surface of Silicon, and that the wafer is insulating. Using boundary conditions for the *total* electric field, find a relation between E_{mi}, E_{mt}, and E_{mr}. Call this equation (1). (**2 Points**)

Solution: Since the tangential component of the electric field is continuous across the air-Silicon interface, the relation is

$$E_{mi} + E_{mr} = E_{mt}. \tag{7}$$

(d) We know that the amplitudes of **H** and **E** are related by the wave impedance of the material medium. Write the incident magnetic field amplitude H_{mi} in terms of E_{mi}, H_{mt} in terms of E_{mt}, and H_{mr} in terms of E_{mr}. Make sure you keep track of the directions of the vectors before writing the scalar equation. (**2 Points**)

Solution: Since the magnetic field intensity and the electric field of an EM wave are related by the local wave impedance, $H_{mi} = E_{mi}/\eta_1$, $H_{mr} = E_{mr}/\eta_1$, and $H_{mt} = E_{mt}/\eta_2$.

(e) Using boundary conditions for the *total* magnetic field intensity, find a new relation between E_{mi}, E_{mt}, and E_{mr} from part (d). Call this equation (2). (**2 Points**)

Solution: Since the tangential component of the magnetic field intensity is also continuous, we have

$$\frac{E_{mi}}{\eta_1} - \frac{E_{mr}}{\eta_1} = \frac{E_{mt}}{\eta_2}. \tag{8}$$

Note that the negative sign is to keep track of the direction of the reflected component of the magnetic field intensity.

(f) Use Equations (1) and (2) from above to find the ratios $T = E_{mt}/E_{mi}$ and $R = E_{mr}/E_{mi}$ in terms of the impedances of air (η_1) and Silicon (η_2). Explain why T and R are related to the transmitted and reflected parts of the EM wave. What values do they take when $\eta_1 = \eta_2$? What does this signify? (**3 Points**)

Solution: Solving Eqs (7 & 8) above, we have

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}, \tag{9}$$

and

$$R = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}. \tag{10}$$

Since T is the ratio of the amplitudes of the electric fields of the transmitted and the incident waves, it represents the part of the EM wave that is transmitted, and similarly, since R is the ratio of the reflected EM wave amplitude and the incident amplitude, it represents the part that is reflected.

When $\eta_1 = \eta_2$, clearly $T = 1$ and $R = 0$, meaning that the EM wave is transmitted entirely (no reflection) if there is no impedance mismatch at the interface. This can only happen if there is no mismatch in the material properties (ϵ_r, μ_r) across the interface, i.e., we have the same material on both sides.

End

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8$ m/s.
Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r\mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Fields : $[\oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot \mathbf{D} = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \mathbf{D} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

In all of the above, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J_c} = \sigma\mathbf{E}$, where σ is the conductivity of the material medium.
In addition to Maxwell's equations above in (B), the boundary conditions are -
For electric fields, $\mathbf{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$, and $\mathbf{n} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$.
For magnetic fields, $\mathbf{n} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$ and $\mathbf{n} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{E} = \hat{E_m}e^{-\hat{\gamma}z}$, where
$\hat{\gamma} = \alpha + j\beta$,
$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2}$,
$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]^{1/2}$,
and the corresponding $\hat{H}$ is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where
$\hat{\eta} = \sqrt{\frac{\mu}{(\epsilon - j\frac{\sigma}{\epsilon})}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1+(\frac{\sigma}{\omega\epsilon})^2]^{1/4}} e^{j\frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})}$.

The skin depth of a conductive medium is $\delta = 1/\alpha$.
The total power stored in the electric field is $\int_v \frac{1}{2}\epsilon|\mathbf{E}|^2 dv$ and in the magnetic field is $\int_v \frac{1}{2}\mu|\mathbf{H}|^2 dv$.
Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.
The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\eta_0}\mathbf{a_z}$ in vacuum or air, and by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\hat{\eta}} e^{-2\alpha z} \cos\theta \mathbf{a_z}$, where $\theta = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta dr d\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta dr d\theta d\phi$.

E) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,

grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_1}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$,

div $\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$, and

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$