
Fundamentals of Electromagnetic Fields and Waves: I

Fall 2007, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term II

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given in the **end** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1 (4 Points)

Answer the following short questions (1 point each):

- (a) Which three parameters describe the behavior of electric and magnetic fields and the propagation of electromagnetic waves inside material medium?
- (b) Of the three material parameters in part (a), only one leads to energy (or power) loss when an EM wave propagates. Which one? Why?
- (c) In order to gather information about outer space, why is it necessary to send out satellites and space telescopes outside the earth's atmosphere?
- (d) What is the only property of an EM wave that does not change when it crosses from one medium to another? Find the change in the wavelength when an EM wave that had $\lambda = 500$ nm in air enters water with $\epsilon_r \approx 81$ and $\mu_r = 1$.

Problem 2 (6 Points)

An infinitely long straight wire is carrying a current given by $I(t) = I_0 \cos \omega t$. A small conducting loop of area $\mathbf{S} = S\hat{\mathbf{n}}$ and resistance R_0 placed at a very large distance ρ_0 from the wire. Find

- (a) The current flowing through this loop as a function of the angle θ the normal to the surface $\hat{\mathbf{n}}$ makes with the axis of the infinite wire, and
- (b) The power (in Watts) picked up by the loop as a function of the angle. Can we connect the small loop to light up a bulb (in principle)? At what angle θ will the bulb glow the brightest if it does?

Continued ...

Problem 3 (10 Points)

An electromagnetic plane wave moves from air (ϵ_{r1}, μ_{r1}) into a wafer of silicon (ϵ_{r2}, μ_{r2}) at normal incidence (Poynting vector along the interface normal), as shown in Figure 1. \mathbf{E}_i , \mathbf{H}_i , and \mathbf{P}_i as shown in the figure stand for the electric field, magnetic field intensity, and the Poynting vector of the *incident* EM wave respectively. The incident electric and magnetic field vectors are parallel to the Silicon-air interface, as shown. Just like for any wave incident on a material medium, a part of it will be transmitted and part of it will be reflected. As shown in the figure, the subscript “*t*” stands for the transmitted components, and “*r*” stands for the reflected components. Answer the following questions -

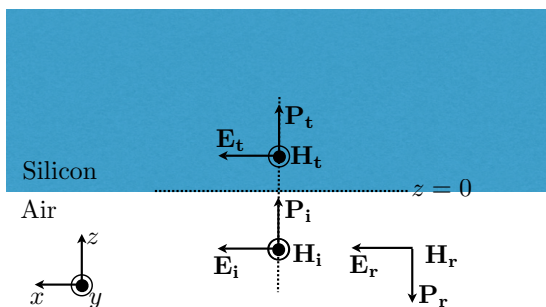


Figure 1: An electromagnetic plane wave moving from air into a silicon wafer.

- Call the wave impedance of air as η_1 . Find the wave impedance η_2 in Silicon. **(0.5 Point)**
- The Poynting vector indicates the direction of wave propagation, as well as the movement of energy. Clearly, the reflected wave carries part of the energy back towards $-z$ (shown in Fig 1). Using this fact, and assuming that the reflected electric field points in the direction indicated, find the direction of the magnetic field of the reflected wave. **(0.5 Point)**
- Now that all directions of the vectors are fixed, lets find the relationships between the fields in air and in Silicon. We can write the incident electric field in the phasor notation as $\hat{\mathbf{E}}_i = E_{mi}e^{-j\beta z}\mathbf{a}_x$ where E_{mi} is the amplitude of the incident electric field. Assume that there are no charges or currents on the surface of Silicon, and that the wafer is insulating. Using boundary conditions for the *total* electric field, find a relation between E_{mi} , E_{mt} , and E_{mr} . Call this equation (1). **(2 Points)**
- We know that the amplitudes of \mathbf{H} and \mathbf{E} are related by the wave impedance of the material medium. Write the incident magnetic field amplitude H_{mi} in terms of E_{mi} , H_{mt} in terms of E_{mt} , and H_{mr} in terms of E_{mr} . Make sure you keep track of the directions of the vectors before writing the scalar equation. **(2 Points)**
- Using boundary conditions for the *total* magnetic field intensity, find a new relation between E_{mi} , E_{mt} , and E_{mr} from part (d). Call this equation (2). **(2 Points)**
- Use Equations (1) and (2) from above to find the ratios $T = E_{mt}/E_{mi}$ and $R = E_{mr}/E_{mi}$ in terms of the impedances of air (η_1) and Silicon (η_2). Explain why T and R are related to the transmitted and reflected parts of the EM wave. What values do they take when $\eta_1 = \eta_2$? What does this signify? **(3 Points)**

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9} \text{F/m}$,

Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7} \text{H/m}$.

Speed of light in free space: $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8 \text{ m/s}$.

Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r\mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Fields : $[\oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot \mathbf{D} = \rho_v]$

Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$

Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$

Ampere's Law : $[\oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \mathbf{D} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}]$

Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

In all of the above, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J}_c = \sigma\mathbf{E}$, where σ is the conductivity of the material medium.

In addition to Maxwell's equations above in (B), the boundary conditions are -

For electric fields, $\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$, and $\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$.

For magnetic fields, $\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$ and $\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{E} = \hat{E}_m e^{-\hat{\gamma}z}$, where

$$\hat{\gamma} = \alpha + j\beta,$$

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} [\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]}^{1/2},$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} [\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]}^{1/2},$$

and the corresponding \hat{H} is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where

$$\hat{\eta} = \sqrt{\frac{\mu}{\epsilon - j\frac{\sigma}{\omega}}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1 + (\frac{\sigma}{\omega\epsilon})^2]^{1/4}} e^{j\frac{1}{2} \tan^{-1}(\frac{\sigma}{\omega\epsilon})}.$$

The skin depth of a conductive medium is $\delta = 1/\alpha$.

The total power stored in the electric field is $\int_v \frac{1}{2}\epsilon|\mathbf{E}|^2 dv$ and in the magnetic field is $\int_v \frac{1}{2}\mu|\mathbf{H}|^2 dv$. Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.

The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P}_{av} = \frac{|E_m|^2}{2\eta_0} \mathbf{a}_z$ in vacuum or air, and by $\mathbf{P}_{av} = \frac{|E_m|^2}{2\hat{\eta}} e^{-2\alpha z} \cos\theta \mathbf{a}_z$, where $\theta = \frac{1}{2} \tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$)

$$d\mathbf{l} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z,$$

$$d\mathbf{s}_x = dydz\mathbf{a}_x, d\mathbf{s}_y = dx dz\mathbf{a}_y, d\mathbf{s}_z = dx dy\mathbf{a}_z,$$

$$dv = dx dy dz.$$

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$)

$$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi d\mathbf{y}\mathbf{a}_\phi + dz\mathbf{a}_z,$$

$$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho, d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi, d\mathbf{s}_z = \rho d\rho d\phi\mathbf{a}_z,$$

$$dv = \rho d\rho d\phi dz.$$

Spherical: (Unit Vectors: $\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$)

$$d\mathbf{l} = dr\mathbf{a}_r + r d\theta\mathbf{a}_\theta + r \sin \theta d\phi\mathbf{a}_\phi,$$

$$d\mathbf{s}_r = r^2 \sin \theta d\theta d\phi\mathbf{a}_r, d\mathbf{s}_\theta = r \sin \theta dr d\phi\mathbf{a}_\theta, d\mathbf{s}_\phi = r dr d\theta\mathbf{a}_\phi,$$

$$dv = r^2 \sin \theta dr d\theta d\phi.$$

E) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a vector field $\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3$,

$$\text{grad } \Phi = \nabla\Phi = \frac{1}{h_1} \frac{\partial\Phi}{\partial u_1} \mathbf{a}_1 + \frac{1}{h_2} \frac{\partial\Phi}{\partial u_2} \mathbf{a}_2 + \frac{1}{h_3} \frac{\partial\Phi}{\partial u_3} \mathbf{a}_3,$$

$$\text{div } \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(A_1 h_2 h_3)}{\partial u_1} + \frac{\partial(A_2 h_1 h_3)}{\partial u_2} + \frac{\partial(A_3 h_1 h_2)}{\partial u_3} \right], \text{ and}$$

$$\text{curl}(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a}_1}{h_2 h_3} & \frac{\mathbf{a}_2}{h_1 h_3} & \frac{\mathbf{a}_3}{h_1 h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, z	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors ($\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$)	$\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$	$\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ , 1	1, r , $r \sin \theta$