Fundamentals of Electromagnetic Fields and Waves: I

Fall 2007, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term Exam

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- Necessary formulae and values of constants are given in the **last page** of this document.
- There are **THREE** problems in this exam. Answer all.

Problem 1

Answer the following short questions.

a) Write down a few reasons why we are studying Electromagnetic Fields and Waves. (**2 Points**)

b) The Divergence theorem states that for a vector field $\mathbf{F}$, the relation $\oint_s \mathbf{F} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{F} dv$ holds, where s is the closed surface enclosing the volume v. However, this relation holds only if $\mathbf{F}$ satisfies certain conditions. What are these conditions? Does the vector $\mathbf{F} = x\mathbf{a_x}$ qualify for application of the divergence theorem for a sphere centered at the origin? (**2 points**)

c) A FM radio station transmits electromagnetic plane waves at a frequency of $f = 100$ MHz. Find the wavelength of the signal. If the maximum electric field strength of the signal is 0.1 V/m, what is the maximum magnetic field for the signal? (**2 Points**)

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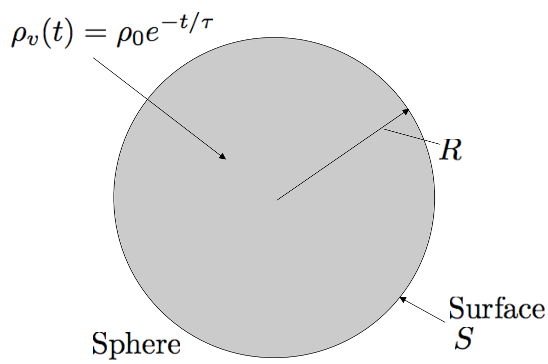

Figure 1: A spherical volume with decaying volume charge.

Problem 2 (5 Points)

A spherical region of radius R is shown in Figure 1. It has a volume charge distribution that is decaying with time according to the relation $\rho_v(t) = \rho_0 e^{-t/\tau}$. Here ρ_0 and τ are constants that characterize the decay process. The volume charge density is uniformly distributed inside the sphere at all times.

a) Use the charge continuity relation (given in the last page) to find the current density vector $\mathbf{J}(t)$ on the surface S of the sphere at time t. Explain each step in the process, and perform any consistency checks (dimensions, etc) for your answer.

b) For $\rho_0 = 1.0$ C/m^3, $\tau = 1$ ns, and $R = 100$ nm, find the magnitude of the total *current* $I(t)$ flowing out of surface S as a function of time t. Sketch its value vs time.

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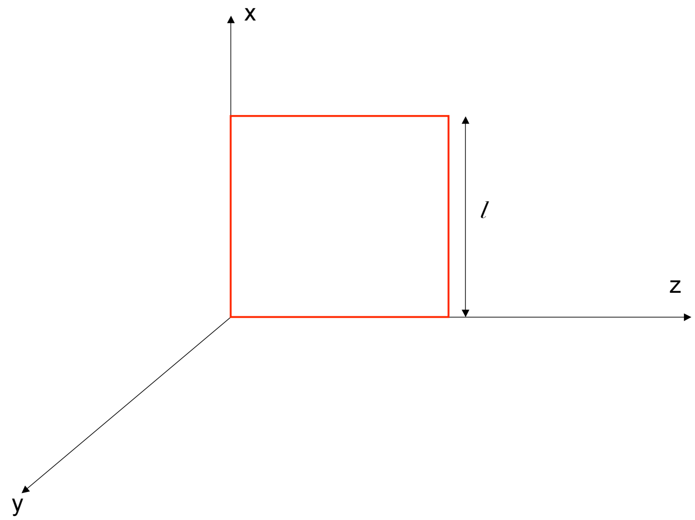

Figure 2: A square loop interacts with an electromagnetic wave.

Problem 3 (**9 Points**)

The electric field vector of an electromagnetic plane wave propagating in free space is given by

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\omega t - \beta_0 z)\mathbf{a_x}$$

a) Identify the wavelength, frequency, polarization, and the direction of propagation of the plane wave.

b) Write down the expression for the corresponding magnetic field $\mathbf{B}(\mathbf{r}, t)$.

c) Refer to Figure 2. A square conductive loop of side l and total resistance R_0 is now placed in the path of the electromagnetic wave as shown. Explain **qualitatively** why the electromagnetic wave will cause a current to flow in the loop.

d) Find the magnitude of the induced current flowing in the loop.
(You may or may not need the following relations:
$\int \sin(az)dz = -\frac{\cos(az)}{a}$+C,
$\int \cos(az)dz = \frac{\sin(az)}{a}$+C,
$\sin(a) - \sin(b) = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})$,
$\cos(a) - \cos(b) = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$)

e) Assume that the length of the loop l is much smaller than the wavelength of the wave (show why this is equivalent to the condition $\beta_0 l \ll 1$). Simplify your answer to part d) using the approximation $\sin(x) \approx x$ for $x \ll 1$. Explain the significance of this simplification.

(List of Formulae in the next page...)

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 3 \times 10^8$ m/s.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Felds : $[\oint_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \epsilon_0 \mathbf{E} \cdot d\mathbf{S})]$, $[\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{d\epsilon_0 \mathbf{E}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

C) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta dr d\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta dr d\theta d\phi$.

D) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,
div $\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$, and

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$