Fundamentals of Electromagnetic Fields and Waves: I

Fall 2007, EE 30348, Electrical Engineering, University of Notre Dame

Final Exam Solutions (30 Points)

- Formulae are given at the end of this questionnaire.
- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- There are **four** questions in this exam. Please attempt all questions.

Problem 1 (10 Points)

Answer the following short questions.

a) (**2 Points**) Can the following vector fields represent a *static* Electric field?

i) $\mathbf{A} = 3\phi \mathbf{a_\phi}$.

Solution: For a vector field to represent a static electric field, Faraday's law dictates that $\nabla \times \mathbf{E} = 0$. $\nabla \times \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(3\phi\rho)\mathbf{a_z} = \frac{3\phi}{\rho}\mathbf{a_z} \neq 0$. Therefore $\mathbf{A}$ cannot represent a static electric field.

ii) $\mathbf{B} = x\mathbf{a_x} + y\mathbf{a_y}$.

Solution: By the same argument, since $\nabla \times \mathbf{B} = 0$, $\mathbf{B}$ can represent a static electric field.

b) (**3 Points**) The electrostatic potential distribution in the region $(x \geq 0, y \geq 0, z \geq 0)$ which is filled with a material of dielectric constant ϵ is given by

$$V(x, y, z) = V_0 e^{-x/L_x} e^{-y/L_y} e^{-z/L_z}$$

in Volts. Find the point (x, y, z) where the volume charge density $\rho_v(x, y, z)$ reaches the maximum magnitude, and the volume charge density $|\rho_v|_{max}$ at that point.

Solution: By Poisson's equation, the volume charge density at any point is related to the electrostatic potential by the relation $\rho_v = -\epsilon\nabla^2 V = -\epsilon(\frac{1}{L_x^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2})V_0 e^{-x/L_x} e^{-y/L_y} e^{-z/L_z}$. Clearly, it reaches a maximum at $(0, 0, 0)$, and the charge density at that point is $|\rho_v|_{max} = \epsilon(\frac{1}{L_x^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2})V_0$.

c) (**2 Points**) The region between two parallel plates of a capacitor is filled with a material of dielectric constant ϵ and conductivity σ. Find the maximum permissible value of the conductivity such that any charge put on the plates at time $t = 0$ can be stored for at least t_1 seconds before reducing to 1/100 of its initial value. As discussed in class, the charge decays exponentially, with the time constant given by the characteristic dielectric relaxation time $\tau_d = \epsilon/\sigma$.

Solution: Since the charge decays as $Q(t) = Q(0)e^{-t/\tau_d}$, it is clear that to meet the requirement, we must have $Q(t_1) = Q(0)e^{-t_1/\tau_d} \geq Q(0)/100 \rightarrow \tau_d \geq t_1/\ln 100 \rightarrow \sigma \leq \frac{\epsilon}{t_1}\ln 100$.

d) (**3 Points**) Show that the skin depth δ for an electromagnetic wave is
i) independent of the frequency of the wave if it propagates in a region of very low conductivity[1], &
ii) goes as the inverse square root of the frequency of the wave ($\delta \propto 1/\sqrt{\omega}$) if it propagates in a region of very high conductivity.

Solution: Since $\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1+(\frac{\sigma}{\omega\epsilon})^2}-1]^{1/2}$, for $(\sigma/\omega\epsilon) << 1$, $\alpha \approx \omega\sqrt{\frac{\mu\epsilon}{2}}[1+\frac{1}{2}(\frac{\sigma}{\omega\epsilon})^2-1]^{1/2} \approx \frac{1}{2}\sqrt{\frac{\mu\sigma^2}{\epsilon}}$. Hence the skin depth for EM wave propagation in a low conductivity material is given by $\delta = \frac{1}{\alpha} \approx \sqrt{\frac{4\epsilon}{\mu\sigma^2}}$, which is clearly not dependent on the frequency ω of the wave.

On the other hand, for a highly conductive medium, $(\sigma/\omega\epsilon) >> 1$, $\alpha \approx \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{(\frac{\sigma}{\omega\epsilon})^2}]^{1/2} \approx \sqrt{\frac{\mu\sigma\omega}{2}}$. Hence the skin depth for EM wave propagation in a low conductivity material is given by $\delta = \frac{1}{\alpha} \approx \sqrt{\frac{2}{\mu\sigma\omega}}$, which goes as $1/\sqrt{\omega}$.

[1] you might need the approximation $\sqrt{1+x} \approx 1 + x/2$ for $x << 1$.

Problem 2 (6 Points)

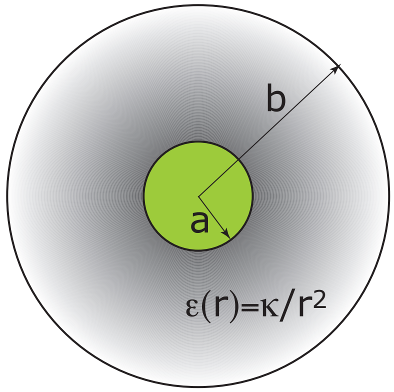

Figure 1: Problem (2)

A spherical capacitor has inner radius a and outer radius b. The region between the two spheres is filled with an *inhomogenous* dielectric medium, with dielectric constant given by $\epsilon(r) = \kappa/r^2$ (see Figure 1). Find the capacitance of the structure. Show all steps.

Solution:

To find the capacitance, assume that we put $+Q$ charge on the inner sphere, and $-Q$ on the outer one. If we can find the voltage difference between the two conductors, the capacitance will be given by $C = Q/V$. The electric flux density $\mathbf{D}$ can be found in the region $a < r < b$ by applying the integral form of Gauss's law: $\oint \mathbf{D} \cdot d\mathbf{S} = Q_{encl} \rightarrow \mathbf{D}(r) = \frac{Q}{4\pi r^2}\mathbf{a_r}$. Then, the electric field is given by $\mathbf{E}(r) = \frac{\mathbf{D}(r)}{\epsilon(r)} = \frac{Q}{4\pi\kappa}\mathbf{a_r}$, using $\epsilon(r) = \kappa/r^2$. Then, the voltage difference between the two conductors is given by $V_{ab} = \int_a^b \mathbf{E}(r) \cdot d\mathbf{l} = \frac{Q}{4\pi\kappa}(b - a)$, and the capacitance is given by $C = Q/V_{ab} = \frac{4\pi\kappa}{b-a}$.

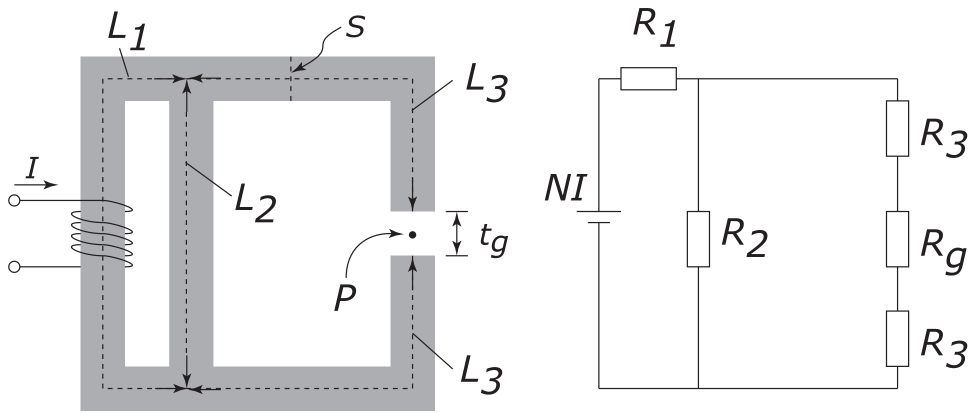

Figure 2: Problem (3)

Problem 3 (**6 Points**)

Refer to the magnetic circuit shown in Figure 3. The ferromagnetic core has a permeability μ, and the geometrical dimensions are shown (the cross-sectional area s is assumed to be uniform throughout the length of the core). You are given a current source which can deliver a current of I Amperes, and a long wire. Design the windings (i.e., how many turns N would you need) to create a magnetic field B_0 at point P in the air gap? Find the answer in terms of the variables given. Neglect any flux leakage from the magnetic core and fringing fields in the air gap.

Solution:

The equivalent electrical circuit is shown in the figure above. The following are the corresponding reluctances: $R_1 = L_1/\mu S$, $R_2 = L_2/\mu S$, $R_3 = L_3/\mu S$, and $R_g = t_g/\mu_0 S$. The magnetomotive force is NI, which will create a net magnetic flux in the branch with the air gap equal to

$$\psi_m^{gap} = B_0 S = \frac{R_2}{R_2 + 2R_3 + R_g} \cdot \frac{NI}{R_1 + R_2||(R_g + 2R_3)}. \tag{1}$$

Here $a||b = (\frac{1}{a} + \frac{1}{b})^{-1}$, the parallel addition of resistors. Clearly, to produce a magnetic field B_0 in the air gap, the number of turns necessary is given by

$$N = \frac{R_2 + 2R_3 + R_g}{R_2} \cdot [R_1 + R_2||(R_g + 2R_3)] \cdot \frac{B_0 S}{I}. \tag{2}$$

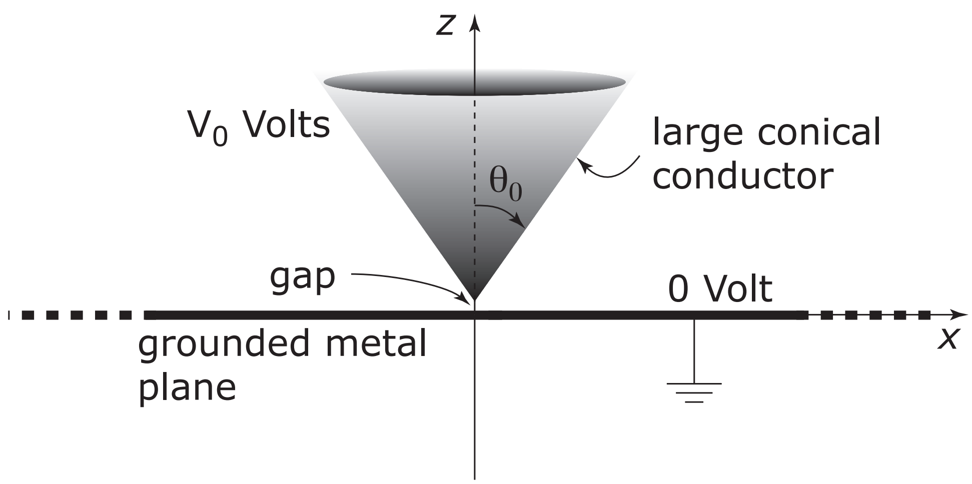

Figure 3: Problem (4)

Problem 4 (**8 Points**)

Refer to Figure 3. A large conducting cone (angle θ_0) is placed on a grounded ($V = 0$ Volt) conducting plane with a tiny gap separating it from the plane. The cone is maintained at a voltage V_0 Volts.

a) Use Laplace's equation to find the electric potential $V(r, \theta, \phi)$ in the region $\theta_0 < \theta < \frac{\pi}{2}$. You might need the following in your calculation:

$\int \frac{d\theta}{\sin\theta} = \ln(\tan\frac{\theta}{2}) + C$, where C is a constant, and

$\tan\frac{\pi}{4} = 1$.

<u>Solution:</u>

Starting with Laplace's equation in spherical coordinates in the region of interest, and making use of the symmetry $[\partial(...)/\partial r = \partial(...)/\partial\phi = 0]$, we get

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\frac{\partial V}{\partial\theta}] = 0. \tag{3}$$

Integrating this once, we get

$$\frac{\partial V}{\partial\theta} = \frac{A}{\sin\theta}, \tag{4}$$

where A is a constant. Integrating one more time, we get

$$V = A\ln[\tan\frac{\theta}{2}] + B, \tag{5}$$

where B is another constant. Using the two boundary conditions - $V(\theta = 0) = 0$ and $V(\theta = \theta_0 = V_0$, we find the two constants, and then the voltage is given by

$$V(r,\theta,\phi) = V(\theta) = V_0 \frac{\ln[\tan\frac{\theta}{2}]}{\ln[\tan\frac{\theta_0}{2}]}. \tag{6}$$

b) Calculate the electric field $\mathbf{E}$ in the region $\theta_0 < \theta < \frac{\pi}{2}$ using your result from part (a).

Solution:

The electric field is given by $\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta$. This leads to

$$\mathbf{E} = -\frac{V_0}{r \sin\theta \ln[\tan\frac{\theta_0}{2}]}\mathbf{a}_\theta, \tag{7}$$

which indicates that the electric field points along the $\mathbf{a}_\theta$ direction from the conical conductor to the planar conductor (remember the ln factor is negative!). The electric field gets stronger as we approach the tip of the cone ($r \to 0$), and weakens as we get farther from the top $|E| \to 0$ as $r \to \infty$. Clearly this implies that there is a charge pileup near the tip of the cone.

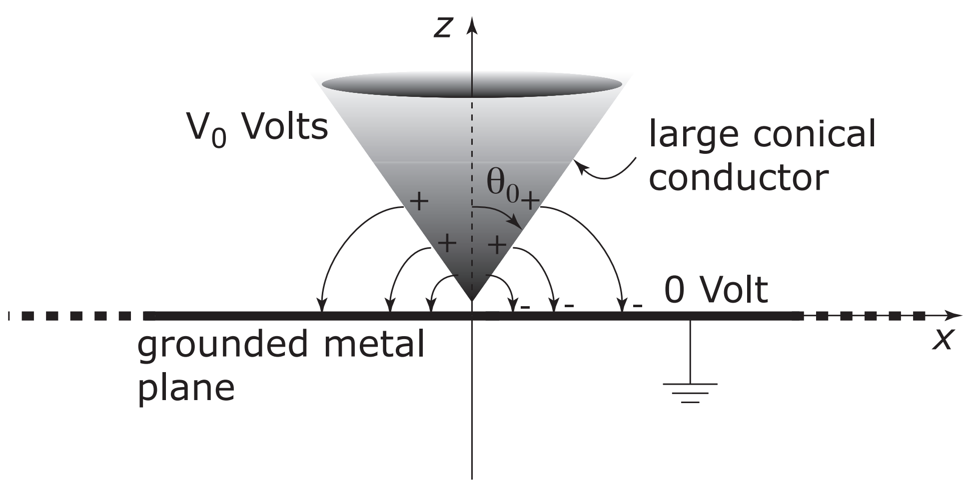

Figure 4: Problem (4)

c) Find the charge density on the two conductors from your result of part (b).

Solution:

The sheet charge density ρ_s on the metallic conductors is given by the boundary condition $\mathbf{D} \cdot \mathbf{n} = \rho_s$. For the conical conductor, the charge is positive since the field lines orignate from there (assuming $V_0 > 0$). It is given by

$$\rho_s^{cone}(r,\theta,\phi) = \epsilon_0 \mathbf{E}(r,\theta=\theta_0,\phi) \cdot \mathbf{a}_\theta = -\frac{\epsilon_0 V_0}{r \sin\theta_0 \ln[\tan\frac{\theta_0}{2}]}, \tag{8}$$

and the corresponding charge on the planar conductor is given by

$$\rho_s^{plane}(r,\theta,\phi) = \epsilon_0 \mathbf{E}(r,\theta = \frac{\pi}{2},\phi) \cdot -\mathbf{a}_\theta = \frac{\epsilon_0 V_0}{r \ln[\tan \frac{\theta_0}{2}]}. \tag{9}$$

The charge and the electric field lines are sketched in Figure 4.

End.
(Formulae in the next page...)

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8$ m/s.
Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r\mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Fields : $[\oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot \mathbf{D} = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \mathbf{D} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

In all of the above, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J_c} = \sigma\mathbf{E}$, where σ is the conductivity of the material medium.
In addition to Maxwell's equations above in (B), the boundary conditions are -
For electric fields, $\mathbf{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$, and $\mathbf{n} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$.
For magnetic fields, $\mathbf{n} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$ and $\mathbf{n} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{E} = \hat{E_m} e^{-\hat{\gamma} z}$, where
$\hat{\gamma} = \alpha + j\beta$,
$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2}$,
$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]^{1/2}$,
and the corresponding $\hat{H}$ is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where
$\hat{\eta} = \sqrt{\frac{\mu}{(\epsilon - j\frac{\sigma}{\epsilon})}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1+(\frac{\sigma}{\omega\epsilon})^2]^{1/4}} e^{j\frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})}$.
The skin depth of a conductive medium is $\delta = 1/\alpha$.

The total energy stored in the electric field is $W_E = \int_v \frac{1}{2}\epsilon|\mathbf{E}|^2 dv$ and in the magnetic field is $W_M = \int_v \frac{1}{2}\mu|\mathbf{H}|^2 dv$, and therefore the total energy stored in a volume with both electric and magnetic fields is given by $W = W_E + W_M$.

Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.
The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\eta_0}\mathbf{a_z}$ in vacuum or air, and by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\hat{\eta}} e^{-2\alpha z}\cos\theta\mathbf{a_z}$, where $\theta = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Static Electric and Magnetic Fields

• Under static conditions, the Electric field is a conservative field, and therefore can be defined as the gradient of a scalar electric potential, i.e.,
$\mathbf{E} = -\nabla V$
and equivalently, the potential difference between two points can be uniquely determined by the line integral of the electric field:
$V_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$.

• The electric potential due to point, line, sheet, and volume charges are given by, respectively, $V_{point} = \frac{Q}{4\pi\epsilon r}$, $V_{line} = \int \frac{\rho_l dl}{4\pi\epsilon r}$, $V_{sheet} = \int \frac{\rho_s ds}{4\pi\epsilon r}$, $V_{vol} = \int \frac{\rho_v dv}{4\pi\epsilon r}$, where l, s, v stand for line, sheet, and volume respectively.

• The electric potential follows the principle of superposition; for example, for N point charges Q_i each located at r_i, the total potential is
$V = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon r_i}$.

• The capacitance of an object is a geometrical property that is defined as the ratio of the positive charge to the resulting potential difference between the conductors, i.e.,
$C = \frac{Q}{V}$.

• Gauss's law for electric field may now be re-cast in terms of the electric potential; this leads to Poisson's and Laplace's equations:
$\nabla^2 V = -\frac{\rho_v}{\epsilon}$ (Poisson's equation), &
$\nabla^2 V = 0$ (Laplace's equation, valid if $\rho_v = 0$).

• A scalar field for the Magnetic field is not possible. However, the Magnetic field $\mathbf{B}$ may be re-cast in terms of a Magnetic vector potential $\mathbf{A}$, such that
$\mathbf{B} = \nabla \times \mathbf{A}$, and
$\mathbf{A} = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')dv'}{R}$, where $\mathbf{J}(\mathbf{r}')$ is the current density which produces the magnetic field.

• Magnetic circuits may easily be solved by using the analogies with electrical circuits. The analogies are :
$V \leftrightarrow NI$ (magnetomotive force),
$I \leftrightarrow \psi_m = B \cdot S$, ($\psi_m$: magnetic flux, B: magnetic field, S: cross-sectional area),
$R \leftrightarrow R = \frac{L}{\mu S}$, L: length of magnetic core, μ: permeability,
and $\sigma \leftrightarrow \mu$.
The analogies are valid as long as the permeability of the core is large enough to prevent substantial flux leakage out of the core.

• The ability of an object to produce magnetic flux in response to the current flowing through it is called the self inductance of the object, and is defined as
$L_{11} = \frac{N_1 \psi_{11}}{I_1}$.
Similarly, the mutual inductance between two conductors is given by
$L_{12} = \frac{N_2 \psi_{12}}{I_1}$.

E) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta dr d\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta dr d\theta d\phi$.

F) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a scalar field Φ,
grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$, and
Laplacian $\Phi = \nabla^2\Phi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3})$.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,
div $\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$, and

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$