
Fundamentals of Electromagnetic Fields and Waves: I

Fall 2007, EE 30348, Electrical Engineering, University of Notre Dame

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Final Exam (30 Points)

- Formulae are given at the end of this questionnaire.
- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.
- There are **four** questions in this exam. Please attempt all questions.

Problem 1 (10 Points)

Answer the following short questions.

a) (**2 Points**) Can the following vector fields represent a *static* Electric field?
i) $\mathbf{A} = 3\phi\mathbf{a}_\phi$.
ii) $\mathbf{B} = x\mathbf{a_x} + y\mathbf{a_y}$.

b) (**3 Points**) The electrostatic potential distribution in the region $(x \geq 0, y \geq 0, z \geq 0)$ which is filled with a material of dielectric constant ϵ is given by

$$V(x,y,z) = V_0 e^{-x/L_x} e^{-y/L_y} e^{-z/L_z}$$

in Volts. Find the point (x,y,z) where the volume charge density $\rho_v(x,y,z)$ reaches the maximum magnitude, and the volume charge density $|\rho_v|_{max}$ at that point.

c) (**2 Points**) The region between two parallel plates of a capacitor is filled with a material of dielectric constant ϵ and conductivity σ. Find the maximum permissible value of the conductivity such that any charge put on the plates at time $t = 0$ can be stored for at least t_1 seconds before reducing to 1/100 of its initial value. As discussed in class, the charge decays exponentially, with the time constant given by the characteristic dielectric relaxation time $\tau_d = \epsilon/\sigma$.

d) (**3 Points**) Show that the skin depth δ for an electromagnetic wave is
i) independent of the frequency of the wave if it propagates in a region of very low conductivity[1], &
ii) goes as the inverse square root of the frequency of the wave ($\delta \propto 1/\sqrt{\omega}$) if it propagates in a region of very high conductivity.

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[1] you might need the approximation $\sqrt{1+x} \approx 1 + x/2$ for $x << 1$.

Problem 2 (6 Points)

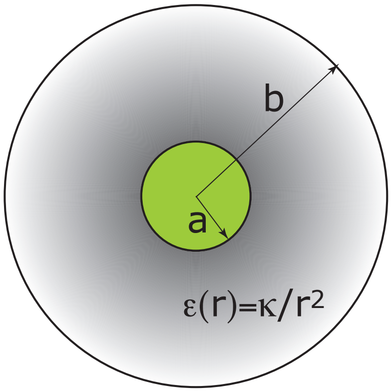

Figure 1: Problem (2)

A spherical capacitor has inner radius a and outer radius b. The region between the two spheres is filled with an *inhomogenous* dielectric medium, with dielectric constant given by $\epsilon(r) = \kappa/r^2$ (see Figure 1). Find the capacitance of the structure. Show all steps.

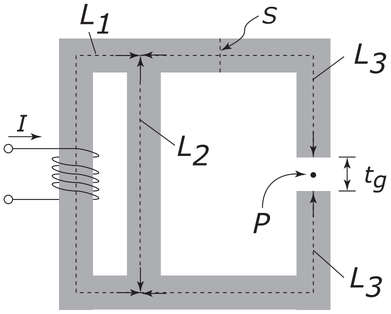

Figure 2: Problem (3)

Problem 3 (6 Points)

Refer to the magnetic circuit shown in Figure 2. The ferromagnetic core has a permeability μ, and the geometrical dimensions are shown (the cross-sectional area s is assumed to be uniform throughout the length of the core). You are given a current source which can deliver a current of I Amperes, and a long wire. Design the windings (i.e., how many turns N would you need) to create a magnetic field B_0 at point P in the air gap? Find the answer in terms of the variables given. Neglect any flux leakage from the magnetic core and fringing fields in the air gap.

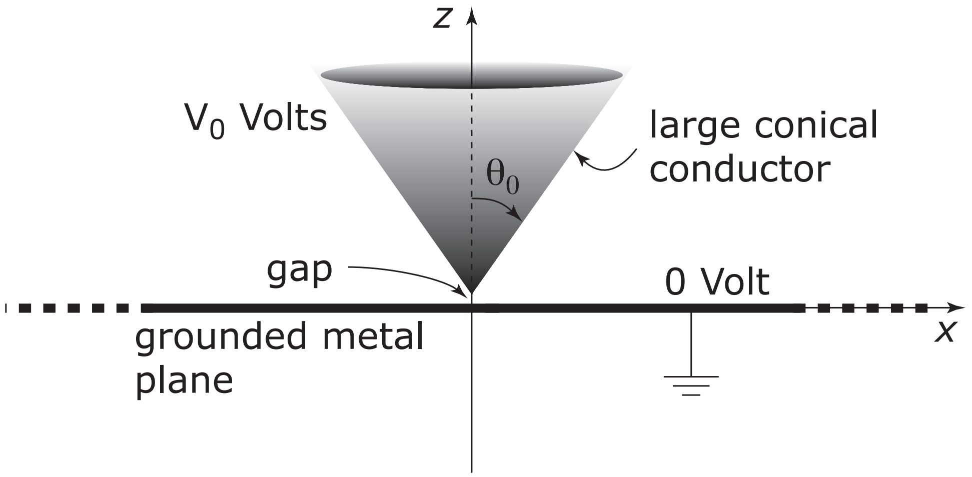

Figure 3: Problem (4)

Problem 4 (8 Points)

Refer to Figure 3. A large conducting cone (angle θ_0) is placed on a grounded ($V = 0$ Volt) conducting plane with a tiny gap separating it from the plane. The cone is maintained at a voltage V_0 Volts.

a) Use Laplace's equation to find the electric potential $V(r, \theta, \phi)$ in the region $\theta_0 < \theta < \frac{\pi}{2}$. You might need the following in your calculation:

$\int \frac{d\theta}{\sin\theta} = \ln(\tan\frac{\theta}{2}) + C$, where C is a constant, and

$\tan\frac{\pi}{4} = 1$.

b) Calculate the electric field $\mathbf{E}$ in the region $\theta_0 < \theta < \frac{\pi}{2}$ using your result from part (a).

c) Find the charge density on the two conductors from your result of part (b).

End.
(Formulae in the next page...)

Formulae

A) Fundamental constants

Permittivity of vacuum: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m,
Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}$H/m.
Speed of light in free space: $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8$ m/s.
Speed of light in non-conductive media: $v = c/\sqrt{\epsilon_r\mu_r}$.

B) Maxwell's Equations: [Integral Form], [Differential Form]

Gauss's Law for Electric Fields : $[\oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv = Q_{encl}]$, $[\nabla \cdot \mathbf{D} = \rho_v]$
Gauss's Law for Magnetic Fields : $[\oint_s \mathbf{B} \cdot d\mathbf{S} = 0]$, $[\nabla \cdot \mathbf{B} = 0]$
Faraday's Law : $[\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}(\int_s \mathbf{B} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}]$
Ampere's Law : $[\oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}(\int_s \mathbf{D} \cdot d\mathbf{S})]$, $[\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}]$
Charge continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$.

In all of the above, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, where ϵ_r is the relative dielectric constant and μ_r is the relative permeability of the material medium.

C) Wave propagation characteristics in material medium

The conduction current density that adds to Ampere's law above is given by $\mathbf{J_c} = \sigma\mathbf{E}$, where σ is the conductivity of the material medium.
In addition to Maxwell's equations above in (B), the boundary conditions are -
For electric fields, $\mathbf{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$, and $\mathbf{n} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$.
For magnetic fields, $\mathbf{n} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$ and $\mathbf{n} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$.

In a generic material medium characterized by the parameters $(\epsilon_r, \mu_r, \sigma)$, the phasor notation of the electric field component of an EM wave moving in the $+z$ direction can be written as $\hat{E} = \hat{E_m}e^{-\hat{\gamma}z}$, where
$\hat{\gamma} = \alpha + j\beta$,
$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} - 1]^{1/2}$,
$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}[\sqrt{1 + (\frac{\sigma}{\omega\epsilon})^2} + 1]^{1/2}$,
and the corresponding $\hat{H}$ is related to the electric field component by the complex impedance $\hat{H} = \hat{E}/\hat{\eta}$, where
$\hat{\eta} = \sqrt{\frac{\mu}{(\epsilon - j\frac{\sigma}{\epsilon})}} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{[1+(\frac{\sigma}{\omega\epsilon})^2]^{1/4}}e^{j\frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})}$.
The skin depth of a conductive medium is $\delta = 1/\alpha$.

The total energy stored in the electric field is $W_E = \int_v \frac{1}{2}\epsilon|\mathbf{E}|^2 dv$ and in the magnetic field is $W_M = \int_v \frac{1}{2}\mu|\mathbf{H}|^2 dv$, and therefore the total energy stored in a volume with both electric and magnetic fields is given by $W = W_E + W_M$.

Power is transported by an EM wave, and the Poynting vector is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.
The time-averaged power transported by an EM wave propagating in the $+z$ direction is given by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\eta_0}\mathbf{a_z}$ in vacuum or air, and by $\mathbf{P_{av}} = \frac{|\hat{E_m}|^2}{2\hat{\eta}}e^{-2\alpha z}\cos\theta\mathbf{a_z}$, where $\theta = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon})$ in a conductive medium.

D) Static Electric and Magnetic Fields

- Under static conditions, the Electric field is a conservative field, and therefore can be defined as the gradient of a scalar electric potential, i.e.,
$\mathbf{E} = -\nabla V$
and equivalently, the potential difference between two points can be uniquely determined by the line integral of the electric field:
$V_{ab} = \int_a^b \mathbf{E}\cdot d\mathbf{l}$.

- The electric potential due to point, line, sheet, and volume charges are given by, respectively, $V_{point} = \frac{Q}{4\pi\epsilon r}$, $V_{line} = \int \frac{\rho_l dl}{4\pi\epsilon r}$, $V_{sheet} = \int \frac{\rho_s ds}{4\pi\epsilon r}$, $V_{vol} = \int \frac{\rho_v dv}{4\pi\epsilon r}$, where l, s, v stand for line, sheet, and volume respectively.

- The electric potential follows the principle of superposition; for example, for N point charges Q_i each located at r_i, the total potential is
$V = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon r_i}$.

- The capacitance of an object is a geometrical property that is defined as the ratio of the positive charge to the resulting potential difference between the conductors, i.e.,
$C = \frac{Q}{V}$.

- Gauss's law for electric field may now be re-cast in terms of the electric potential; this leads to Poisson's and Laplace's equations:
$\nabla^2 V = -\frac{\rho_v}{\epsilon}$ (Poisson's equation), &
$\nabla^2 V = 0$ (Laplace's equation, valid if $\rho_v = 0$).

- A scalar field for the Magnetic field is not possible. However, the Magnetic field $\mathbf{B}$ may be re-cast in terms of a Magnetic vector potential $\mathbf{A}$, such that
$\mathbf{B} = \nabla \times \mathbf{A}$, and
$\mathbf{A} = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')dv'}{R}$, where $\mathbf{J}(\mathbf{r}')$ is the current density which produces the magnetic field.

- Magnetic circuits may easily be solved by using the analogies with electrical circuits. The analogies are :
$V \leftrightarrow NI$ (magnetomotive force),
$I \leftrightarrow \psi_m = B \cdot S$, ($\psi_m$: magnetic flux, B: magnetic field, S: cross-sectional area),
$R \leftrightarrow R = \frac{L}{\mu S}$, L: length of magnetic core, μ: permeability,
and $\sigma \leftrightarrow \mu$.
The analogies are valid as long as the permeability of the core is large enough to prevent substantial flux leakage out of the core.

- The ability of an object to produce magnetic flux in response to the current flowing through it is called the self inductance of the object, and is defined as
$L_{11} = \frac{N_1 \psi_{11}}{I_1}$.
Similarly, the mutual inductance between two conductors is given by
$L_{12} = \frac{N_2 \psi_{12}}{I_1}$.

E) Line, Surface, and Volume vector differential elements

Rectangular: (Unit Vectors: $\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$)
$d\mathbf{l} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$,
$d\mathbf{s_x} = dydz\mathbf{a_x}$, $d\mathbf{s_y} = dxdz\mathbf{a_y}$, $d\mathbf{s_z} = dxdy\mathbf{a_z}$,
$dv = dxdydz$.

Cylindrical: (Unit Vectors: $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$)
$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi dy\mathbf{a}_\phi + dz\mathbf{a_z}$,
$d\mathbf{s}_\rho = \rho d\phi dz\mathbf{a}_\rho$, $d\mathbf{s}_\phi = d\rho dz\mathbf{a}_\phi$, $d\mathbf{s_z} = \rho d\rho d\phi\mathbf{a_z}$,
$dv = \rho d\rho d\phi dz$.

Spherical: (Unit Vectors: $\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$)
$d\mathbf{l} = dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$,
$d\mathbf{s_r} = r^2\sin\theta d\theta d\phi\mathbf{a_r}$, $d\mathbf{s}_\theta = r\sin\theta drd\phi\mathbf{a}_\theta$, $d\mathbf{s}_\phi = rdrd\theta\mathbf{a}_\phi$,
$dv = r^2\sin\theta drd\theta d\phi$.

F) div, grad, & curl expressions in various coordinate systems

Use the metric coefficients from the Table below.

For a scalar field Φ,
grad $\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{a_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{a_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{a_3}$, and
Laplacian $\Phi = \nabla^2\Phi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3})$.

For a vector field $\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}$,
div $\mathbf{A} = \frac{1}{h_1h_2h_3}[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_1h_3)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}]$, and

$$curl(\mathbf{A}) = \begin{vmatrix} \frac{\mathbf{a_1}}{h_2h_3} & \frac{\mathbf{a_2}}{h_1h_3} & \frac{\mathbf{a_3}}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

	Cartesian	Cylindrical	Spherical
Independent Variables (u_1, u_2, u_3)	x, y, x	ρ, ϕ, z	r, θ, ϕ
Vector components (A_1, A_2, A_3)	A_x, A_y, A_z	A_ρ, A_ϕ, A_z	A_r, A_θ, A_ϕ
Unit Vectors $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$\mathbf{a_x}, \mathbf{a_x}, \mathbf{a_x}$	$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a_z}$	$\mathbf{a_r}, \mathbf{a}_\theta, \mathbf{a}_\phi$
Metric Coefficients (h_1, h_2, h_3)	1, 1, 1	1, ρ, 1	1, $r, r\sin\theta$