Fundamentals of Electromagnetic Fields and Waves: I

Fall 2005, EE 30348, Electrical Engineering, University of Notre Dame

Mid Term Exam Solutions

Please show your steps clearly and sketch figures wherever necessary. Points will be awarded for correct steps shown in the solutions.

Fundamental constants: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m, $\mu_0 = 4\pi \times 10^{-7}$H/m.

Problem 1

Answer the following short questions.

a) (1 Point) What fundamental law is violated if displacement current is not included in Maxwell's equations?

Soln:

Neglecting the $\frac{\partial(\epsilon_0 \mathbf{E})}{\partial t}$ term in Ampere's law leads to $\nabla \times (\mathbf{B}/\mu_0) = \mathbf{J}$. Taking the divergence of both sides, we get[1] $\nabla \cdot \nabla \times \mathbf{B}/\mu_0 = 0 = \nabla \cdot \mathbf{J}$, which violates the charge-continuity law which states that $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$ should always be satisfied.

b) (1 Point) Can the vector $\mathbf{A} = 3x\mathbf{a_x} + 2y\mathbf{a_y} - 5z\mathbf{a_z}$ represent a magnetic field?

Soln:

Since $\nabla \cdot \mathbf{A} = 0$, $\mathbf{A}$ can indeed represent a magnetic field.

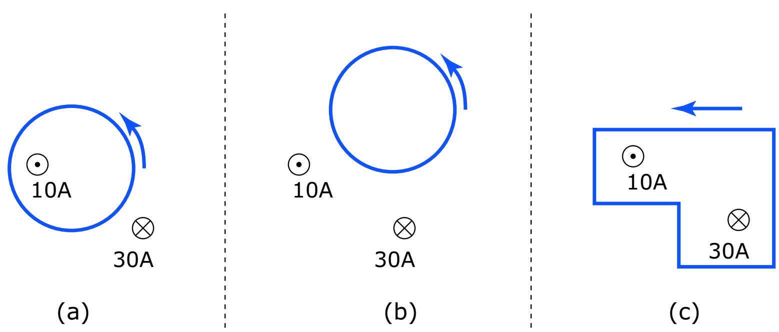

Figure 1: Problem 1 c)

c) (3 Points) Refer to Figure 1 for this part. Evaluate the line integrals $\oint_c (\mathbf{B}/\mu_0) \cdot d\mathbf{l}$ for the three cases, where the directions of the lines in blue are indicated by arrows.

Soln:

[1]Since div[curl(...)] is always zero.

Use the integral form of Ampere's law. Note that there are no time-varying quantities, so the displacement current term is zero. Therefore, Ampere's law reduces to $\oint_c (\mathbf{B}/\mu_0) \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S} = I$, where the direction of the surface is given by the right-hand rule. Therefore, the answers are a) 10A, b) 0 A, and c) (10-30) A = -20 A.

d) (3 Points) The electric field intensity of a uniform plane wave is given by

$$\mathbf{E} = 20\cos(\pi \times 10^9 t - \pi z)\mathbf{a_x}$$

in V/m. Find the direction of propagation, the frequency, and the wavelength of the plane wave. Also find the direction and magnitude of the magnetic field.

Soln:

The direction of propagation is along +ve $z-$direction. The frequency can be found from $\omega = 10^9\pi$ rad/s $= 2\pi f \Rightarrow f = 5 \times 10^8$Hz. Since $\beta_0 = \pi$ m$^{-1} = 2\pi/\lambda$, $\lambda = 2$m. (However... Read the footnote[2]!). The wave propagates along $+z$, and the electric field points in the $+x$ direction. Since $\mathbf{E} \times \mathbf{B}$ gives the direction of propagation, $\mathbf{B}$ is along $\mathbf{a_y}$, and $|B_y| = |E_x|/c$.

Problem 2 (6 Points)

Refer to Figure 2. A square loop of side a is placed near an infinitely long wire carrying a current I. The loop has a resistance R_0. The loop moves with a velocity v_0 away from the wire as shown. Find the current that flows through the loop as a function of the distance ρ from the infinite wire.

Soln:

The magnetic field at a distance ρ from the wire axis is $B = \mu_0 I/2\pi\rho$. Therefore, the magnetic flux through the loop is (See Figure 2)

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{S} = \int_\rho^{\rho+a} \underbrace{(\frac{\mu_0 I}{2\pi\rho'})\mathbf{a}_\phi}_{\mathbf{B}} \cdot \underbrace{(ad\rho')\mathbf{a}_\phi}_{d\mathbf{S}} = \frac{\mu_0 I a}{2\pi}[ln(\rho + a) - ln(\rho)]. \quad (1)$$

From the integral form of Faraday's law, the rate of change of magnetic flux is the emf, or voltage generated. Therefore, the current through the loop is given by

$$I_{loop} = \frac{-d(\Phi)/dt}{R_0} = \frac{\mu_0 I a v_0}{2\pi R_0}[\frac{1}{\rho} - \frac{1}{\rho + a}] = \frac{\mu_0 I a^2 v_0}{2\pi R_0 \rho(\rho + a)}. \quad (2)$$

Here, we have used the chain rule for differentiation in the form $d(ln\rho)/dt = (d(ln\rho)/d\rho) \cdot (d\rho/dt)$, and the fact that $d\rho/dt = v_0$, as given. Check that the final answer is dimensionally correct: dimensions are in Amps on both sides of (2).

Problem 3 (6 Points)

Refer to Figure 2. Two infinitely large sheets carry currents with linear current densities $\pm J_0\mathbf{a_y}$ in A/m respectively, as shown. Find the magnetic field $\mathbf{B}$ this generates between the sheets, and outside them. Indicate the direction of the field, and plot its magnitude as a function of z.

[2] NOTE: There is something fishy in this question - if you use $c = f\lambda$, you get a different wavelength! If you can hunt out the root of this problem, I will award you extra points towards your first mid-term score. Hint: There might be an error in the question itself, as it stands...

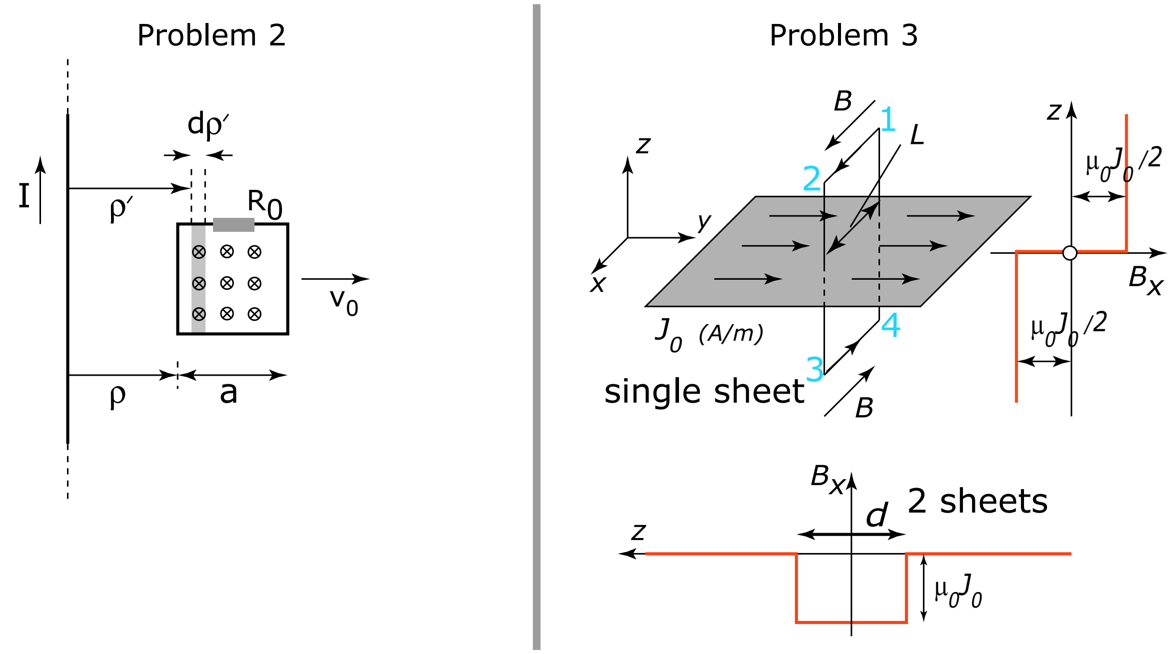

Figure 2: Problems 2 & 3.

(Hint: It might be easier if you solve the problem for one sheet first, and then use the superposition principle. Apply symmetry arguments and the right hand rule, and use Ampere's law in the integral form by choosing a reasonable contour.)

Soln:

To exploit the symmetry of the problem, we should use the steady state Ampere's law in the integral form - $\oint_c \mathbf{B}/\mu_0 \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{S}$, which can be re-written as $\oint_c \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$, I being the total current passing through a chosen contour c.

Choose the contour as shown in Figure 2 for just one current sheet - a rectangle with the top and bottom lengths L. Say the sheet is at $z = 0$. By using the right-hand rule, the magnetic field above the sheet should point in the $+x$ direction, and along $-x$ direction below the sheet. By symmetry, the field cannot change as long as we are at the same distance from the sheet, so the fields are constant in the paths $1 \rightarrow 2$ and $3 \rightarrow 4$. Again, by symmetry, the magnetic field in the $z-$direction is zero everywhere, including paths $2 \rightarrow 3$ and $4 \rightarrow 1$. If the length of the path $1 \rightarrow 2$ is L, the total current passing through the loop is $\int_s \mathbf{J} \cdot d\mathbf{S} = I = J_0 L$ (in Amps).

Therefore, using Ampere's law, we get $\oint_c \mathbf{B} \cdot d\mathbf{l} = \oint_1^2 \mathbf{B} \cdot d\mathbf{l} + \oint_3^4 \mathbf{B} \cdot d\mathbf{l} = 2BL = \mu_0 J_0 L$, which gives the magnitude of the magnetic field directly as $B = \mu_0 J_0/2$, pointing in the $+\mathbf{a_x}$ direction for $z \geq 0$, and in the $-\mathbf{a_x}$ direction for $z < 0$.

Using superposition when both sheets are at a distance d apart, the fields cancel outside the sheets, and add between them. The resulting field is $\mathbf{B} = -\mu_0 J_0 \mathbf{a_x}$ for $-d \leq z \leq 0$, and 0 elsewhere, as shown in the plot.

Note that it is possible to find the field using Biot-Savart law, but that method is far more involved for this case than Ampere's law, which allows us to exploit the symmetry.