Fundamentals of Electromagnetic Fields and Waves: I

Fall 2005, EE 30348, Electrical Engineering, University of Notre Dame

Final Exam: SOLUTIONS

- Please show your steps clearly and sketch figures wherever necessary.
- Points will be awarded for correct steps shown in the solutions.

Fundamental constants: $\epsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$F/m, $\mu_0 = 4\pi \times 10^{-7}$H/m.

Problem 1

Answer the following short questions.

a) (**1 Point**) A fictitious insulating material has a dielectric constant of $\epsilon = 4\epsilon_0$ and permittivity of $\mu = 9\mu_0$. What is the velocity of an electromagnetic wave in the material?

Answer:

The velocity is given by $v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{9\mu_0 \times 4\epsilon_0}} = c/6 = 5 \times 10^7$ m/s.

b) (**2 Points**) The electric field intensity of a uniform plane wave in a conductive medium is given by

$$\mathbf{E}(z,t) = 151e^{-0.435z}\cos(2\pi \times 10^6 t - 0.435z + 0.41)\mathbf{a_x}$$

in V/m. Find the frequency, the wavelength, the phase velocity , and the skin depth of the conductive medium for the plane wave.

Answer:

$f = \omega/2\pi = 1$MHz, $\lambda = 2\pi/\beta = 14.4$m, $v_p = \omega/\beta = 1.44 \times 10^7$m/s, and $\delta = 1/\alpha = 2.3$m.

c) (**1 Point**) Two small metal spheres are kept 4 meters apart in air. One of them is connected to a battery of +4 volts, and the other is connected to ground. Find the line integral $\int_c \mathbf{E} \cdot d\mathbf{l}$ for two contours: one a straight line connecting the spheres, and the second a semi-circle of radius 2 meters connecting the spheres. Do not worry about the sign of your answer.

Answer:

The line integral evaluates to <u>4 Volts</u>, irrespective of the path.

d) (**2 Points**) The electrostatic potential distribution in a region of dielectric constant ϵ is given by

$$\Phi(x,y,z) = V_0 \sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L})\sin(\frac{\pi z}{L})$$

in Volts. Find the volume charge density $\rho_v(x,y,z)$ in that space.

Answer:

Using Poisson's equation, $\rho_v = -\epsilon\nabla^2\Phi$, we get directly that $\rho_v(x,y,z) = 3\epsilon(\pi/L)^2\Phi(x,y,z)$.

Problem 2 (**5 Points**)

Consider a parallel plate capacitor, with the plates of area A and the separation between them d. Let the region between the plates be filled with an insulator of dielectric constant ϵ. If a voltage V is applied across the plates, show that the electrostatic energy stored in the capacitor is $\frac{1}{2}CV^2$. Use the result that the electrostatic energy stored in a volume is given by $W_E = \frac{1}{2}\int_{vol} \mathbf{E}\cdot\mathbf{D}dv$. Neglect all fringing fields.

Answer:

The electric field between the plates is $\mathbf{E} = \frac{V}{d}\mathbf{a_z}$, and the displacement vector is $\mathbf{D} = \epsilon\mathbf{E}$. Therefore, the electrostatic energy stored in the capacitor is given by

$$W_E = \frac{1}{2}\int_{vol} \mathbf{D}\cdot\mathbf{E}dv = \frac{1}{2}\epsilon(\frac{V}{d})^2 \times A \times d = \frac{1}{2}(\epsilon\frac{V}{d})V^2 = \frac{1}{2}CV^2,$$

since the capacitance is $C = \epsilon\frac{A}{d}$.

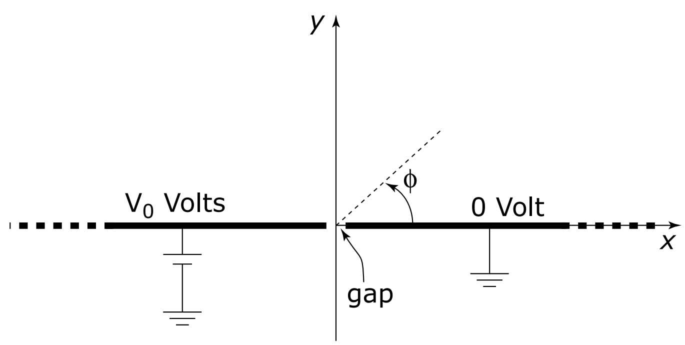

Figure 1: Problem (3)

Problem 3 (**8 Points**)

Refer to Figure 1. Two infinite conducting planes (extending over $-\infty \leq z \leq +\infty$) are separated by a small air gap. The plane on the right ($x > 0$) is kept at a potential $\Phi_1 = 0$ Volt, and that on the left ($x < 0$) is kept at a potential $\Phi_2 = V_0$ Volts. The charge density for the region $0 < \phi < \pi$ as well as $\pi < \phi < 2\pi$ is zero (ϕ is the angle in cylindrical coordinates, as shown in the figure).

(a) Solve Laplace's equation in the region $0 < \phi < \pi$, and find the electrostatic potential $\Phi(\phi)$ in this region. Look up the expressions for any operators you need in the handout.

(b) Use your result in part (a) to find the Electric Field $\mathbf{E}$ in the region $0 < \phi < \pi$. Sketch some representative field lines.

(c) Calculate the charge density on each conducting plate.

Answer

a) To solve Laplace's equation $\nabla^2\Phi = 0$ in the region $0 < \phi < \pi$, we first note that the problem has cylindrical symmetry. Hence, the Laplacian is chosen in the cylindrical co-ordinate system. Further, we note that the potential does not depend on the co-ordinates (ρ, z). Hence, Laplace's equation reduces to

$$\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

We can write the solution as $\Phi(\phi) = A\phi + B$, which obviously satisfies the equation. With the boundary conditions $\Phi(\phi = 0) = 0$ Volt, we get $B = 0$. With the boundary condition $\Phi(\phi = \pi)$, we get $A = V_0/\pi$. Therefore, the electrostatic potential for $0 < \phi < \pi$ is given by

$$\Phi(\rho, \phi, z) = \frac{V_0}{\pi}\phi.$$

In a similar fashion, we find the electrostatic potential in the region $\pi < \phi < 2\pi$ to be

$$\Phi(\rho, \phi, z) = V_0(2 - \frac{\phi}{\pi}).$$

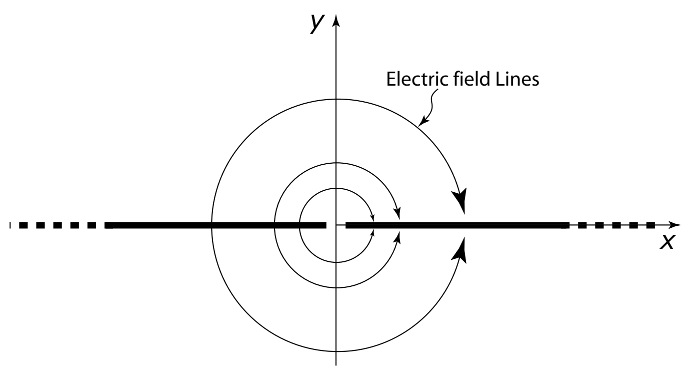

Figure 2: Problem (3) Solution

(b) To find the electric field, we use

$$\mathbf{E} = -\nabla\Phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\mathbf{a}_\phi.$$

We get, for $0 < \phi < \pi$, $\mathbf{E} = -\frac{V_0}{\pi\rho}\mathbf{a}_\phi$. Similarly, for $\pi < \phi < 2\pi$, we get the electric field to be $\mathbf{E} = +\frac{V_0}{\pi\rho}\mathbf{a}_\phi$. The field lines are sketched in Figure 2.

(c) The surface charge density can be found from the discontinuity of the electric field by using the boundary condition $\rho_s = \epsilon_0(E_{1n} - E_{2n})$. On the plate $x > 0$, the charge is obviously negative since electric field lines terminate on it. The surface charge density on this plate is $\rho_s(x) = -\frac{2\epsilon V_0}{\pi x}$, i.e., it varies as $\rho_s \sim 1/x$. By symmetry, the charge density on the plate $x < 0$ is positive, and is given by $\rho_s(x) = +\frac{2\epsilon V_0}{\pi x}$. It also varies as $\rho_s \sim 1/x$. Check that the units of the surface charge density is indeed C/m^2.

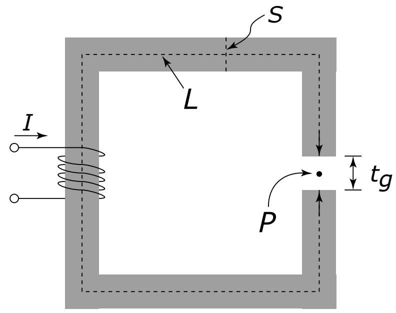

Figure 3: Problem (4)

Problem 4 (**6 Points**)

Refer to the magnetic circuit shown in Figure 3. The ferromagnetic core has a permeability μ, and the geometrical dimensions are shown (the cross-sectional area s is assumed to be uniform throughout the length of the core). You are given a current source which can deliver a current of I Amperes, and a long wire. Design the windings (i.e., how many turns N would you need) to create a magnetic field B_0 at point P in the air gap? Find the answer in terms of the variables given. Neglect any flux leakage from the magnetic core and fringing fields in the air gap.

Answer:

By drawing an equivalent magnetic circuit with reluctances $\Re_L = L/\mu s$ and $\Re_{gap} = t_g/\mu_0 s$, and using "Ohm's" law for the circuit, we get directly that the number of turns required is given by

$$N = \frac{B_0}{I}\left(\frac{L}{\mu} + \frac{t_g}{\mu_0}\right).$$