

$|\varphi_n\rangle$ in terms of the $\{|\psi_j^k\rangle\}$ basis. The appropriate state assigned to the ensemble E in the context C_k will be the following:

$w_k = \sum_j w_j^k |\psi_j^k\rangle\langle\psi_j^k|$, where $w_j^k = \sum_n w_n |c_j^n|^2$, that is:

$$\begin{aligned} w_k &= \sum_j (\sum_n w_n |c_j^n|^2) |\psi_j^k\rangle\langle\psi_j^k| \\ &= \sum_{j,n} w_n |c_j^n|^2 |\psi_j^k\rangle\langle\psi_j^k|. \end{aligned}$$

We now have the following:

$$\begin{aligned} \text{Tr}(W_{qm}^M S) &= \text{Tr}[(\sum_n w_n |\varphi_n\rangle\langle\varphi_n|) (\sum_j a_j^s |\psi_j^k\rangle\langle\psi_j^k|)] \\ &= \text{Tr}\{[\sum_n w_n (\sum_m c_m^n |\psi_m^k\rangle) (\sum_r c_r^{n*} \langle\psi_r^k|)] (\sum_j a_j^s |\psi_j^k\rangle\langle\psi_j^k|)\} \\ &= \text{Tr}(\sum_{n,m,r,j} a_j^s w_n c_m^n c_r^{n*} |\psi_m^k\rangle\langle\psi_r^k| |\psi_j^k\rangle\langle\psi_j^k|) \\ &= \text{Tr}(\sum_{n,m,r} a_r^s w_n c_m^n c_r^{n*} |\psi_m^k\rangle\langle\psi_r^k|) \\ &= \sum_{n,m,r,t} a_r^s w_n c_m^n c_r^{n*} \langle\psi_t^k | \psi_m^k\rangle \langle\psi_r^k | \psi_t^k\rangle \\ &= \sum_{n,t} a_t^s w_n |c_t^n|^2, \end{aligned}$$