KANT, LOGICISM, AND THE NATURE OF MATHEMATICS

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1

Early in the last century, the logicists and logical positivsts attacked Immanuel Kant's philosophy of space and time by attacking his arguments in the Transcendental Aesthetic of the first *Critique*. As interpreted by Rolf Horstmann[1], there are two general lines of attack, one made by Hans Reichenbach and the other by Bertrand Russell. Both attack Kant's philosophy of geometry, summed up by the thesis that the truths of geometry are truths of space and, simultaneously, synthetic *a priori*. The readings of the Aesthetic favored by the Analytic tradition see the four proofs that space is a pure *a priori* intuition as either failing completely or as presupposing the 'argument from geometry' of the Transcendental Exposition of the Concept of Space. This puts all the burden on the Transcendental Exposition, which in turn is read so that it depends on geometric truths *qua* spatial truths being synthetic *a priori*. The difference in the two lines of attack comes in the next and final step. For Reichenbach, spatial truths are strictly empirical, and hence geometric truths are at best empirical, and not *a priori*. For Russell, on the other hand, the proofs of theorems of geometry are logical derivations from basic geometric concepts, which in turn are defined using only basic logical concepts, and are synthetic only in the sense that logic is synthetic. In Russell's own words,

> the abstract logical method, based upon the logic of relations . . . [is] still adequate, and enable[s] us to define all the classes of entities which mathematicians call spaces, and to deduce from the definitions all the propositions of the corresponding Geometries.[2]

This is a direct and negative response to what Russell identifies as 'the [question] of chief importance to us, as regards the Kantian theory . . .

[1] Horstmann, 1976
[2] Russell, 1903/1938, 461

are the reasonings in mathematics in any way different from those of Formal Logic?'[3].

There are several tactics a partisan of an at least vaguely Kantian view might use to respond to these anti-Kantian early Analytic arguments, two of which are employed by Horstmann. These are to argue that, first, this reading of the Aesthetic is flawed, so that the four arguments Kant gives in the Metaphysical Exposition do not need the synthetic apriority of geometry to work; and, second, concluding that geometry is not synthetic *a priori* does not mean we need to abandon the conclusions of the Metaphysical Exposition. Solid steps in this direction have been made by Horstmann, as well as Lisa Shabel[4], Daniel Warren[5], and Emily Carson[6]. In this paper, I wish to consider a third and more specific response to the Russellian argument that one might make, which Horstmann may not have brought up simply because it runs against most of mainstream philosophy of mathematics of the past century: denying the final step of the logicist argument. More specifically, I want to question whether the logicist reductions of mathematical definitions, propositions, and inferences are really adequate in the way Russell used that term in the quotation above, ie, whether these reductions can really support the claim that geometric reasoning has been shown to be analytic or strictly logical.

I shall proceed not by challenging the adequacy of particular formal systems, eg, pointing out that conventional first-order logic has problems with counterfactuals, or that ZFC (standard first-order axioms for set theory) cannot deal with classes in ways mathematicians are perfectly comfortable with. Neither will I consider whether the axioms of a particular formal system are genuinely analytic. My aim is broader and deeper, to question the *possibility* of an adequate representation of mathematics in *any* formal system, and thereby stress that the lack of arguments for this claim is a major weakness of logicism.

First, however, let me describe what I mean by logicism, since I will be using the term in a wider sense than usual. Conventionally, logicism refers to the perspective on mathematical knowledge held by such philosophers as Gottlob Frege, Russell, and WVO Quine, and already articulated above: that all mathematics can be reduced to formal derivations from some basic logical principles (probably including at least some set-theoretic notions under 'logic'). But, for the purposes of this paper, I also include the formalist views commonly associated

[3]Russell, 1903/1938, 457
[4]Shabel, 2003
[5]Warren, 1998
[6]Carson, 1997

with David Hilbert and Haskell Curry (though this is something of a caricature in the case of Hilbert), that mathematics can be reduced to inferences from arbitrarily assumed axioms according to a system of syntactical manipulations (possibly exempting some basic principles of arithmetic that let the syntactical system itself be treated mathematically). Common to both views is the idea that mathematical inferences can and should be undertaken independent of the meanings of mathematical terms: either strictly universal methods of reasoning are to be used (logicism), or reasoning is reduced to bare calculation (formalism). *Contra* Kant, a mathematician does not require an *a priori* synthetic connection between some sort of intuitive experience of mathematical objects and the concepts predicated of those objects, once the axioms are stated. Hence, mathematics, and geometry in particular, is analytic.

2

Consider the following mathematical system: a group of permutations G of a set X of 54 elements, such that G is generated by three sets of three permutations each. Each of the nine generators has order 4; in each set, there are two permutations which act on 20 elements of X, and one which acts on 12, so that the elements one permutation acts on are fixed by the other two in the same set, and two elements are fixed by all three in that set. Using group theoretic notation, we can say that the subgroup of G generated by each set is isomorphic to

$$\left\langle x, y, z \mid xy = yx, xz = zx, yz = zy, x^4 = y^4 = z^4 = 1 \right\rangle .$$

Finally, to this we add approximately 45 other relations between the nine generators, many of which will be several hundred letters long; I omit them only to save space.

Now, given such a G and X (with the generators of G written explicitly as permutations of X), one can be asked a variety of questions. We will restrict our attention to one in particular, whether some permutation on X is in G or not. We can think of this as a game: I give you a permutation, and you have to tell me whether or not it is in G. We can also add an additional task, to write the permutation explicitly in terms of the generators.

Of course, almost anyone presented with this problem in exactly the form above (with the additional relations given explicitly, of course) would immediately wonder why they should bother with what will obviously be a tedious, difficult if not impossible, and seemingly pointless exercise. The system does not become significant until it is given a specific referent, which in this case is a Rubik's cube: the 54 elements represent the small squares that form the 6 faces of the cube, and the permutations in G all possible configurations of the squares from the nine basic rotations of the parts of the cube, three in each plane. The approximately 49 relations abstract the way the basic rotations interact. The problem in the previous paragraph is simply to solve the Rubik's cube, ie, perform (list) the basic rotations needed to put the cube in a given permutation (solving it then involves reversing these steps).

I call this the Rubik's cube analogy, and appeal to it to justify the claim that simply giving a representation of several salient features of a being does not justify the ontic or epistemic elimination of that being in favor of the representation. We may be able to use the representation as an aid to reasoning about the referent – the specific instance of G pertaining to the Rubik's cube can be used to understand the interactions of the basic rotations and develop patterns used to solve the cube, for example – but without the referent, the representation can be too obtuse to work with. Hence, it does not seem too radical to claim that epistemic access to the representation may require prior epistemic access to the referent. Indeed, mathematicians studying the Rubik's cube and other such puzzles usually move back and forth between group-theoretic reasoning and drawing conclusions based on how the puzzles are played. Knowledge concerning G does not supplant knowledge of the Rubik's cube.

This example leads to concerns about both Russell- and Hilbert-style reductions of mathematics: the logicists should either defend their proposed epistemic reduction, or provide an alternative defence of their program. One such non-epistemological response might be that they are uncovering the 'true nature' of the mathematical beings in question, which was heretofore lost or confused when mathematicians abandoned rigour[7] while considering the foundations of their discipline. A contemporary logicist would probably not make such a claim, but this was the attitude Frege took in the *Grundlagen*, indeed, the *raison d'etre* for that essay: 'Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle.'[8] Frege, of course, agreed with Kant on geometry, and considered it to be synthetic *a priori*; but we can suppose a different logicist might claim that their reduction laid out the 'true nature' of geometrical objects as strictly logical.

[7]Taking a particularly logicist definition of rigour for granted; more on this below.
[8]Frege, 1884/1950, 5

The problem is that this hypothetical response is exactly that: strictly hypothetical. I can find no arguments of this sort in logicists after Frege. The situation is even worse with the response later logicists are more likely to make: that they are offering, for the first time, a rigorous definition of terms previously used only in an informal way; this is the attitude Russell seems to have taken in *Principles*. Here the logicists must defend their potential reduction with a careful analysis of the way the informal terms are used by mathematicians, and argue that their reduction captures whatever is found to be essential to those terms. Russell's unelaborated assertion of adequacy is, in a word, inadequate.

The closest logicists come to an adequacy argument are claims that their formal systems 'prove' all the same theorems as, well, other formal systems. But even if some did claim their formal system of geometry 'proved the same theorems' as geometry done by ordinary mathematicians, this is only an adequacy claim if mathematical beings are determined entirely by the location of mathematical terms in proofs; that is, if logicism is correct. I am not aware of any positive arguments for the adequacy of logicist reductions that do not take this form.

3

In the last section, I argued that logicist reductions of mathematical terms require substantial adequacy claims. In this section, I wish to argue that logicist reductions of mathematical inferences require similar justification. Let me lay out a logicist account of mathematical rigour, following Michael Detlefsen[9]:

(1) lapses of rigour occur when information about the subject under consideration is concealed;
(2) the risk of concealing information in a proof is under adequate control when and only when all the information contained in it is forced into its premises;
(3) that which is logical in character is information-less;
(4) hence, the information contained in a proof is forced into its premises by making all of its inferences logical in character, ie, by 'logicizing' it;
(5) and therefore the danger of lapses of rigour is under adequate control when and only when a proof is 'logicized'.

(3) might initially seem somewhat contentious – on the most obvious reading of 'information', Russell himself changed his mind about (3) several times. But what Detlefsen has in mind with this use of 'information' are content-specific or topical considerations: (3) simply

[9]Detlefsen, 1993, 279

asserts that logical inferences do not depend on the subject-matter of the antecedent and consequent in an inference, merely the form of the statement; which should not be a controversial claim. This conception of rigour can also be characterized as emphasizing the relationship between the truth-conditions of two propositions: a rigorous inference from one statement to another shows that the truth-conditions of the first are a subset of the truth-conditions of another. If 'truth-conditions' are defined using Tarski's definition for formal systems, all of this is completely compatible with my broad use of 'logicism'.

Now, I wish to press against (2) by outlining three alternative approaches to mathematical rigour and inference which reject it, following Detlefsen. In particular, all three claim that true rigour comes from incorporating information about the mathematical beings under consideration at every step in a proof, not by pushing it back into the axioms. My purpose is not to present these as superior alternatives to logicism (although I do admit I think they are); rather, simply by their existence, these alternatives show that logicism cannot take its definition of rigour and its model of inference for granted. An ingenious formal system shows nothing if mathematical knowledge cannot be formalized.

The first alternative comes from Poincaré, who presented an apparent paradox for epistemologies of mathematics:

> The very possibility of the science of mathematics seems an insoluble contradiction. If this science is deductive only in appearance, whence does it derive that perfect rigor no one dreams of doubting? If, on the contrary, all the propositions it enunciates can be deduced one from another by the rules of formal logic, why is not mathematics reduced to an immense tautology? ... Shall we then admit that the enunciations of all those theorems which fill so many volumes are nothing but devious ways of saying A is A?[10]

Suppose Russell is right, and formal logic is adequate for all mathematical reasoning; then, following (3), it seems odd to say mathematical results are all that significant; and yet results in pure mathematics are, in fact, widely regarded as significant. To resolve this, Poincaré offers an alternative approach to mathematical rigour, characterized by Detlefsen[11] by a pair of principles, Epistemic Typification and Epistemic Conservation.

[10]Poincaré, 1946, 31

[11]Detlefsen, 1990, 508 and Detlefsen, 1992, 351

Epistemic Typification simply asserts that knowledge should be categorized by cognitive modes, eg (nonexhaustively), mathematical knowledge, empirical knowledge, or logical knowledge. Epistemic Conservation adds that there are salient features for inferences of each type of knowledge, which limit the means to achieve new knowledge. Empirical knowledge, for example, has very few salient features and, as a compromise between a desire for absolute certainty and the limitations of sense data, there are relatively few restrictions on the types of inferential moves one may make to get new empirical knowledge. Mathematical knowledge, on the other hand, has a much higher standard: for Poincaré, a proof of a given conclusion from some premises can only count as genuinely mathematical knowledge if the premises, the conclusion, *and* the type of inference itself count as mathematical knowledge.[12] Inferences which count as mathematical knowledge may also be logical knowledge, eg, modus ponens, but being logical knowledge is neither necessary nor sufficient to be mathematical knowledge; Poincaré's most famous example is mathematical induction, which is perfectly acceptable as mathematical knowledge but, capturing 'an infinity of syllogisms'[13], is not logical knowledge. (Hence Russell's insistence that induction is merely an axiom which characterizes the sequence of natural numbers, not a principle of inference; had he accepted this, clearly arithmetical inference could not be grounded in universal forms of inference.)

It should be clear that Poincaré is rejecting (2) exactly as I explained above: for him, genuine mathematical rigour requires careful attention to the mathematical subject matter, and, even more, neglecting the content of a mathematical inference, ie, treating it as a logical inference, actually results in a *loss* of rigour.

LEJ Brouwer's intuitionism is next. I mean Brouwer specifically, and not intuitionism as it has been construed by the Analytic tradition, especially Heyting and Dummett; the difference will be clear below. Brouwer's epistemology is built on two 'Acts'; in the first, he

> completely separat[es] mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time.[14]

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[12]Ibid

[13]Poincaré, 1946, 38

[14]Brouwer, 1951, 4

Hence, Brouwer asserts that mathematical inference is not about establishing relationships of inference between propositions – that is, a linguistic endeavor – but rather an action of mathematical intuition or experience – and hence prior to linguistic activity, at least in regards to arithmetic and the continuum[15]. Indeed, on his account, logical rules of inference, far from being the basic forms of mathematical reasoning, are nothing more than codifications of perceived regularities in the linguistic accounts of mathematical activity; hence formal logic is actually epistemically derivative upon genuine mathematical reasoning, not the other way around.

Brouwer's intuitionism is thus a much more radical rejection of logicism's approach to the epistemology of mathematics than Poincaré's epistemic principles, and far more radical than the formal constructivist semantics usually identified with intuitionism: Poincaré's approach can be cashed out in terms of propositions and inferences, but Brouwer believes a genuine epistemology of mathematics must begin with the experience of doing mathematics, with mental actions epistemically prior to language. This is clearly incompatible with the conventional understanding of intuitionism's main critiques of conventional mathematics, which reads Brouwer as merely rejecting the equivalence of certain formal sentences. Detlefsen gives a Brouwerian account of rigour: those proof-activities are rigourous

> which are . . . those of the mind at perfect (causal-manipulatory) rest, with no designs on causal dominion over nature or even over one's own stream of inner mathematical experience,[16]

a sort of mathematical quietism, wherein the mathematician allows mathematical beings to be revealed in themselves.[17] Brouwer then rejects (2) in much the same way Poincaré does: rigour is lost when mathematical information about the situation at hand is deliberately

[15]Detlefsen, 1990, 515ff

[16]Detlefsen, 1990, 524

[17]Put in these terms, one is lead to consider a number of striking parallels between Brouwer and Martin Heidegger's philosophy: a phenomenological approach to truth, scathing critiques of logic and a general scepticism of the ability of language to accurately reflect reality, along with this call for quietistic respect for the autonomy of one's subject of inquiry. On the other hand, Heidegger's critique of logic extends to a critique of 'mathematisation', which might conflict with the methods Brouwer uses to develop intuitionism. Unfortunately, further consideration is beyond the scope of the present paper.

concealed as it is forced into a (necessarily inadequate) linguistic expression. Here mathematics depends on the subjective experiences of a knower for its certainty and rigour, not the removal of those sentences.

Third and finally, Detlefsen generalizes the attitudes of Brouwer and Poincaréin what he calls a 'Kantian' approach[18]. For a Kantian mathematician, mathematical knowledge consists of having an 'intuition' (the exact details of which are left deliberately vague, to accommodate both Brouwer and Poincaré) of the epistemic transition from knowledge of one mathematical situation to another, not demonstrating that the truth-conditions of the antecedent are a subset of those of the consequent as in the logicist definition of proof: a Kantian proof consists of mathematical knowledge 'which, under the action of the inferrer's cognition, leads to mathematical knowledge of the conclusion even though the truth-conditions of the former may not cover those of the latter'[19]. And Kantian mathematical rigour is the principle that 'an inference is gapless when one grasps a mathematical architecture in which a *mathematical* justification of the conclusion is seen as a development of a *mathematical* justification of the premisses'[20]. Detlefsen elaborates on the contrast with the logicist conception of rigour, explicitly rejecting what I have labelled (1):

> The elimination of gaps is therefore no longer to be identified with the *exclusion* of topic-specific information in an inference ... but rather with the *inclusion* of a mathematical insight which transforms a proof of the premiss into a proof of the conclusion.[21]

The reader may balk at the way this epistemology is given using the vague term 'mathematical justification'. I would suggest this wariness comes from the present dominance of logicism in framing the terms and standards for contemporary philosophy of mathematics; the best way for the reader to understand Detlefsen's definitions is with a naturalistic reading: an argument, a proposed proof in particular, is a mathematical justification simply if it is accepted as such by the community of mathematicians.

It should be stressed that this is not a call for some kind of constructivism, which can be cashed out as some particular formal deductive, or even semantic, system ('$A \to B$ means I can turn a construction for A into a construction for B'); such inferences would no

[18]Detlefsen, 1993, 281ff
[19]Detlefsen, 1993, 281
[20]Detlefsen, 1993, 284, my emphasis
[21]Detlefsen, 1993, 284

longer have the particularly mathematical character this epistemology requires, namely, the attention to the content of the statements in question, contra (1). All three alternatives have discussed shift the epistemological focus to the experience of a mathematical knower coming to have some mathematical knowledge, and therefore involve at least a partial move away from considerations simply of sentential forms.

This is also why these epistemologies can offer adequacy arguments logicism cannot, or at least not as easily: proofs published in journals and relayed in seminars do not consist of a thousand applications of modus ponens, but can be read as a procession of steps the reader or hearer should undertake to come to see why the theorem is true, to experience the truth of the theorem. The Kantian begins with a focus on mathematical knowers and their epistemic relationship to theorems and proofs: the Kantian is doing epistemology of mathematics. The logicist's starting point, by contrast, is the truth-conditional relationships between propositions in general; in the past century, logicians have become quite ingenious in constructing formal systems that 'do the work' of mathematical ones, but the experience of the working mathematician cannot be dealt with by proposing some new scheme of formal deduction. Logicists must respond to these epistemologies with arguments about the nature of mathematical inference, if they wish to defend their epistemology with anything resembling intellectual honesty and good-will.

4

However, I do not think logicism can actually offer such a defence, at least not in a way that will be at all convincing to those who already find the alternatives of the last section appealing, and I believe this is due to a more fundamental disagreement over the nature and rôle of logic in thought. For the logicist, logic is a tool for inference: one can take a statement or collection of statements, deduce further statements using the tools logic provides, and we are then justified in stating that we have knowledge of these conclusions (though, the logicist should concede, many of these conclusions will not constitute any sort of substantial proper extension of our knowledge). Note that this is consistent with (3), the logicist view that logical inference is information-less, in the sense that logical inferences do not depend on the content of the sentences under consideration; but it does not seem to run up against the assumptions underlying Poincaré's concern that, if mathematical inference is merely logical inference, new mathematical knowledge cannot constitute a substantial proper extension of our knowledge.

This is because Poincaré takes a very different view of logic. Following Kant, Poincaré takes logic to be only a canon of reason, not an organon: concordance with logical rules is a *necessary* condition for the validity of a *given* argument, but logical rules are insufficient for declaring an argument valid, and logic alone can only suggest a possible argument from assumptions to conclusion, not give one. As Kant puts it:

> [Logical rules] concern only the form of truth, ie, of thinking in general, and are to that extent entirely correct but not sufficient. For although a cognition may be in complete accord with logical form, ie, not contradict itself, yet it can still always contradict the object. The merely logical criterion of truth, namely the agreement of a cognition with the general and formal laws of understanding and reason, is therefore certainly the *conditio sine qua non* and thus the negative condition of all truth; further, however, logic cannot go, and the error that concerns not form but content cannot be discovered by any touchstone of logic.[22]

Kant goes on to identify this illegitimate use of logic as organon as one of the primary sources of paradox in the philosophy of his day, and indeed one of the fundamental points of the first *Critique* is that pure reason, including logic, cannot go beyond a close connection with objects as given in experience (*Gegenstände*). This is precisely why the three alternatives to logicism given in the last section can be considered Kantian: they insist on attention to the connection between knowledge of mathematics and the content of that knowledge.

There is one obvious attack a logicist might make against the Kantian epistemologies, which is useful for illustrating the distinction I am making here. Since Descartes, and particularly in the contemporary fields of algebraic topology and algebraic geometry, mathematical reasoning related to space is not necessarily all that 'spatial', and does not seem to involve intuition in the 'pictures in the mind' sense as consistently as the Kantian epistemologies would seem to require. Indeed, the point of introducing algebraic tools into the mathematics of space is to replace difficult or impossibly intractable pictures with discrete calculation, and significant geometric and topological results are

[22]Kant, 1998, A59/B84

proven using algebraic tools[23]. This is the negative logicist claim: the Kantian epistemology cannot handle this 'algebraization' of geometry. Then follows an analogical positive logicist claim: if geometry can be algebraized, that is, spatial reasoning reduced to symbolic manipulations, there is no reason why it cannot be logicized, and reduced to a particular form of symbolic manipulations.

The Kantian responds by pointing out a crucial difference: algebra is mathematical, but the sort of completely contentless symbolic manipulations the logicist has in mind are not (or, if they are, the logicist has simply reduced mathematics to mathematics). The algebraization of geometry does not abandon the mathematical beings under consideration, but instead shifts to a different point of view: the figure, previous given as a system of spatial relations, is still in mind, but now as a system of numerical relations. Hence algebraic reason is still mathematical, and still obeys the Kantian requirement of conformity to objects. By contrast, the logicized geometry would no longer be mathematical – that's the entire point of the reduction, after all – and would instead be pure reason, neglecting the required connection to objects of possible experience. The Kantian view must recognize broad notions of mathematics and intuition, so that both algebra and geometry are accommodated, but it can easily do this while maintaining a distinction between the mathematical and the universal.

So, it should now be obvious why the challenge that, eg, the Poincaréan epistemologist presents to logicism is much more daunting than a mere question of adequacy: rejecting the use of logic as an organon of knowledge means logicism is incoherent from the start. Even questions like the consistency of some proposed neo-Fregean system that attempts to demonstrate that geometry is analytic are meaningless on this view. And, even though logic is accepted as an organon among most contemporary philosophers of math, this is no reason to consider the Kantian views vanquished. Indeed, it seems we must consider Russell's attack on Kant's philosophy of mathematics to be not just incomplete, as the Rubik's cube analogy suggests, but likely a non-starter – unless, of course, he gives a detailed analysis of the place of logic in reason

[23]The best 'simple' example of this is the proof that an arbitrary angle cannot be trisected using a straightedge and compass. Straightedge and compass constructions are shown to correspond to algebraic constructions called finite field extensions, with certain additional restrictions, which in turn correspond to solutions of certain forms of equations; a particular angle is chosen so that trisecting it corresponds to a solution of an equation which cannot be of the particular form, and so we have a proof by contradiction.

and hence a much broader critique of Kant's overall epistemological project, not just matters of mathematics or space.

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