FICTIONALISM AND NATURALISM IN MATHEMATICS

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1. Introduction

The question of our ontological commitments to mathematicals – the question whether numbers, functions, sets, &c, exist, and whether or not it is legitimate to use terms that purport to refer to such entities or variables which purport to range over them[1] – is the question of mathematical realism. Broadly, any position which answers this question in the affirmative is identified as (mathematical) *realism* or *platonism*; and any position which answers in the negative is identified as (mathematical) *anti-realism*, *fictionalism*, or *nominalism*. Realists often give an argument somewhat like the following:

(1) Our ontological commitments are the *prima facie* ontological commitments of our best scientific theory.
(2) The *prima facie* ontological commitments of our best scientific theory include mathematicals.
(3) Hence, our ontological commitments include mathematicals.

Such an argument is called an *indispensability argument*, according as the second premiss says that mathematicals are indispensable to our best scientific theory. The prominence of indispensability arguments within contemporary Anglophone philosophy of mathematics is usually credited to Quine[2] and Putnam[3].

The first premiss of this argument is a statement of at least one kind of *naturalism*. This is a position whose prominence is again usually credited to Quine. In 1955, he wrote the *locus classicus* of this sort of naturalism, 'Posits and reality':

> Having noted that man has no evidence for the existence of bodies beyond the fact that their assumption helps him organize experience

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[1]Field, 1980, 1
[2]Quine, 1976, et al
[3]Putnam, 1979, et al

> [into theory], we should have done well . . . to conclude: such, then, at bottom, is what evidence is, both for ordinary bodies and for molecules [and for mathematicals].[4]

Contemporary philosophers often characterise naturalism as an attitude of philosophical modesty, humility, or subordination to the natural sciences: 'if philosophy conflicts with successful [scientific] practice, the philosophy must give'[5]. As a more ontological – and less epistemological – thesis, it is nicely captured by Burgess as 'a willingness to repeat in one's philosophical moments what one says in one's scientific moments, not taking it back, explaining it away, or otherwise apologizing for it: what we say in our scientific moments is *all right*'[6]. Naturalist philosophers uncritically (or almost uncritically) accept all the *prima facie* ontological commitments in the language scientists actually use.

Hence, in (1), 'our best scientific theory' means 'our best scientific theory *according to the standards of science*', and not according to the standards of 'first philosophy' or an *a priori* epistemology. In 'Posits', Quine identifies five such virtues or standards: simplicity (which seems to encompass both ontological parsimony and elegance of inferences), explanatory power, theoretical unity (the ability of a general picture to account for a variety of empirical phenomena), fecundity (the ability of the theory to serve as the basis for a robust and interesting research programme), and empirical adequacy.[7] Similarly, in her work on naturalism[8], Maddy identifies virtues usually taken by scientists and mathematicians to characterise a good mathematical theory, including applicability and something she calls 'maximize' (on which more in §4.2).

Some may find this sort of naturalism too restrictive. The indispensability argument is still valid if the first premiss is replaced with

(1′) Our ontological commitments *include all* the *prima facie* ontological commitments of our best scientific theory.

[4]Quine, 1976, 251
[5]Maddy, 1997, 171
[6]Burgess, 2004, 19, his emphasis
[7]Quine, 1976, 247; the terminology I have used here is that used by contemporary philosophers of science, not Quine's own.
[8]Maddy, 1997, et al

To say that our ontological commitments include all the *prima facie* ontological commitments of our best scientific theory does not entail that our ontological commitments include *only* or are *exhausted by* the *prima facie* ontological commitments of our best scientific theory. That is, we allow the possibility that we have some ontological commitments beyond those entities mentioned by our best scientific theory, eg, non-physical minds. For our purposes here, it does not matter whether we take (1) or (1′) as our definition of naturalism: I see no reason to think the existence of mathematicals is not independent of the existence of disembodied minds.[9]

Burgess[10] has argued that there are two kinds of fictionalist projects, *hermeneutic* and *revolutionary*, and that both are incompatible with naturalism. Leng[11], responding to Burgess, claims that this is not so, that revolutionary fictionalism as exemplified by Field's programme[12] – the most prominent fictionalist programme of the past quarter-century – is entirely compatible with naturalism. In this paper, I will examine this claim of Leng's: Can a fictionalist give an argument for her or his position which is compatible with naturalism? I begin by briefly considering one possible definition of fictionalism, eventually identifying a definition that does (or would, if realised) seem to be compatible with naturalism. Indeed, I argue that it is (or would be) sufficiently similar to an argument of Quine's for the axiom of constructibility, so that either both are compatible with naturalism or both are not. I then conclude by identifying two serious problems for this position.

2. Defining fictionalism

Both Leng and Field characterise fictionalist projects in terms of a gap between truth and 'goodness', the latter being an epistemological notion: 'truth isn't required for goodness (so necessary truth isn't required either)'[13]; 'the ... fictionalist does not advocate the abandonment of any successful mathematical theory, but rather to account for the success of that theory under the assumption that that theory

[9]If the reader finds this an absurd use of the term 'naturalism', she is invited to replace it with any term she finds more appropriate throughout the paper.
[10]Op cit
[11]Leng, 2005
[12]Field, 1980, et al
[13]Field, 1989, 4

might not in fact be literally true'[14]. This would suggest defining fictionalism as the following claim:

Fictionalism: Platonistic theories are 'good' but not true.

(Here a platonistic theory is one which includes reference to mathematicals; this is contrasted with a nominalistic theory, which does not include reference to mathematicals.) The primary fictionalist task, on this definition, is to identify and explicate this notion of 'good'. Field, for example, argues that a platonistic theory that is conservative with respect to a nominalistic theory[15] 'might be useful in facilitating inferences (between nominalistic premises and nominalistic conclusions)'[16] and hence epistemically valuable without being true. For example[17], consider the premisses

(1) There are exactly twenty-one aardvarks.

(2) On each aardvark there are exactly three bugs.

(3) Each bug is on exactly one aardvark.

and the conclusion

(4) There are exactly sixty-three bugs.

An argument from 1-3 to 4 could be cast in the nominalistic first-order 'theory of spatial relations between macroscopic animals', but it would be extremely long and tedious (and 1 and 4 might not even be human-readable). By contrast, using some 'bridge principles' between these sentences and elementary number theory, there is an extremely simple argument from 1-3 to 4. Then, given that elementary number theory is conservative with respect to the theory of spatial relations between macroscopic animals, we know that the nominalistic inference from 1-3 to 4 is valid without having to work through the tedious nominalistic proof. Hence, elementary number theory has facilitated the inference from 1-3 to 4.

This is not yet an argument for fictionalism – elementary number theory being conservative and useful for facilitating nominalistic inferences does not preclude it

[14]Leng, 2005, 283

[15]A theory $N + P$ is conservative with respect to a theory N if and only if, for all sentences φ in the language of N, if $N + P$ entails φ, then N entails φ. Here 'entails' is ambiguous between logical or semantical or model-theoretic entailment, on the one hand, and syntactical or formal entailment, on the other; cf Shapiro, 1983.

[16]Field, 1989, 59, his parentheses

[17]Cf ibid, 63

being true. But, even if it did, this version of fictionalism is not compatible with naturalism. Recall Burgess' characterisation, *supra*: if naturalists, we make no apology for our apparent references to mathematicals. By contrast, this version of fictionalism says that our apparent references to mathematicals are in need of apology. Or, a bit more precisely, this version of fictionalism is the claim that our ontological commitments do *not* include the *prima facie* ontological commitments of our best scientific theory. Instead, our ontological commitments come from the *prima facie* ontological commitments of a nominalistic theory which is far inferior with respect to the simplicity and elegance of its inferences.

Let us try again.

Fictionalism: Platonistic theories are not our best scientific theories.

This is inspired by Field's claim that the positive content of fictionalism is that 'it must be [at least] possible . . . to develop physics without any appeal to mathematical entities'[18].

What is the force of 'possible' here? Craig Elimination[19] implies that there is an entirely straightforward way to construct a nominalistic theory out of any fairly well-behaved platonistic theory. Hence, if 'possible' means 'logically possible', then it seems that fictionalism is almost automatically true. But Field goes on to say that this claim 'is a very substantial claim; there are serious difficulties in defending it'[20].

Thus, instead of logical possibility, I think what Field means to claim here is that it is possible to formulate a nominalistic theory that is a serious epistemological rival to the ordinary, platonistic theory. As in our simple or simplistic example of bugs and aardvarks, the nominalistic theory will fall short of the platonistic theory in certain respects (elegance of inferences) but do much better in others (especially

[18]Field, 1989, 6

[19]Briefly, the Craig Elimination Theorem says the following: Let $\mathcal{T}$ ('platonistic theory') be a consistent, deductively closed set of sentences in a first-order formal language whose alphabet of non-logical symbols divides into classes N ('nominalistic') and P ('platonistic'), ie, the alphabet for $\mathcal{T}$ is $N \cup P$. Let $\mathcal{T}_N$ ('nominalistic subtheory') be the subset of $\mathcal{T}$ whose alphabet is N. Then, if $\mathcal{T}_N$ is a recursively enumerable set (which is true if $\mathcal{T}$ has a primitive recursive axiomatization and N is recursively enumerable, ie, we can use an ideal computer to generate a list of the sentences in $\mathcal{T}_N$), there is a primitively recursive axiomatization of it.

[20]Ibid; cf Field, 1980, 8

ontological parsimony). Going further, it will not just be a serious rival, but indeed will be evaluated better overall – the nominalistic theory will actually be our best theory, according to the ordinary standards of scientists. Hence 'possible' means 'logically possible', but 'develop physics' must be read as 'develop a serious physics exemplifying the epistemic virtues even better than our currently accepted theories'; and this project seems adequately expressed in the definition above.

Note the way this version of fictionalism responds to the indispensability argument: not by rejecting the first premiss, but by rejecting the second. This fictionalist argues that mathematicals do not show up among the ontological commitments of our best scientific theory; instead, our best scientific theory is mathematical-free.

This version of fictionalism is compatible with naturalism. Indeed, if this claim is accompanied by a particular nominalistic rival to the platonistic status quo, it can readily serve as a 'fictionalist response to the Quinean line, based not on heavy-duty metaphysical principles, but rather on more down-to-earth thoughts regarding the nature of evidence'[21].

3. Fictionalism and $V = L$

With this definition of fictionalism – hencefore, 'naturalist fictionalism' – now in hand, I argue that, not only is it compatible with naturalism, but it greatly resembles a somewhat fictionalist position of Quine's. In particular, the naturalist fictionalist argument is indeed naturalist if, and only if, Quine's argument for $V = L$ is naturalist.

$V = L$ is a candidate axiom for ordinary pure set theory – that is, an axiom which may or may not be added to the standard ZFC axioms for set theory. V is the proper class of all sets, sometimes called the set-theoretic universe. L is the class of *constructible sets*. For my purposes here, L is best characterised by an informal construction using transfinite recursion.

[21]Leng, 2005, 289

For any set X, let

$$\mathrm{Def}(X) = \big\{\{y \in X \mid \exists x_1, \ldots, x_n \in X\ \varphi(y, x_1, \ldots, x_n)\}$$
$$\mid \varphi \text{ is a first-order formula in the language of sets with } n+1 \text{ free variables}\big\}$$

Intuitively, $\mathrm{Def}(X)$ is the set of first-order definable subsets of X. Then, for any ordinal α and limit ordinal λ, set the following definitions:

$$L_0 = \{\}$$
$$L_{\alpha+1} = \mathrm{Def}(L_\alpha)$$
$$L_\lambda = \cup_{\alpha<\lambda} L_\alpha$$
$$L = \cup_\alpha L_\alpha$$

Thus, intuitively, L is built by explicitly constructing its members in discrete stages. $V = L$ therefore asserts that all sets can be constructed in this way. Among other claims, both the Continuum Hypothesis and the Axiom of Choice are provable in the set theory $ZF + V = L$. Rival axioms assert the existence of large cardinals and other sets which cannot be constructed in this way out of smaller sets.

Quine's argument for $V = L$ is, more accurately, an argument that $V = L$ should be accepted as a standard axiom by set theorists. Before presenting the argument, it will be useful to present its context.

> What now of those parts of mathematics that share no empirical meaning, because of never getting applied in natural science? What of the higher reaches of set theory? We see them as meaningful because they are couched in the same grammar and vocabulary that generate the applied parts of mathematics. We are just sparing ourselves the unnatural gerrymandering of grammar that would be needed to exclude them.[22]

We can read this as the indispensability argument for mathematical realism, albeit rather abbreviated: mathematicals play an indispensable rôle in making our best theory elegant, &c, and hence we have an ontological commitment to mathematicals.

[22] Quine, 1992, 94

But not all the mathematicals with which mathematicians might appear to busy themselves:

> The main axioms of set theory are generalities operative already in the applicable part of the domain [ie, in physics]. Further sentences such as as the continuum hypothesis and the axiom of choice, which are independent of those axioms, can still be submitted to the considerations of simplicity, economy, and naturalness that contribute to the molding of scientific theories generally. Such considerations support Gödel's axiom of constructibility, 'V=L'. It inactivates the more gratuitous flights of higher set theory .... *[This, and similar proposals] are of a piece with the simplifications and economies that are hailed as progress within natural science itself.*[23]

I read this as follows:

(1) Non-constructible sets are not indispensable to our best scientific theory, and hence we have no ontological commitment to them.
(2) Indeed, the ordinary standards of good science counsel us to actively deny the existence of non-constructible sets.

The first step is compatible with a mathematical agnosticism: perhaps non-constructible sets exist, perhaps they do not. The second step goes further, arguing for a mathematical atheism: non-constructible sets do not exist. Even if we restrict ourselves to the agnostic first step[24], notice that this particular instance of the indispensability argument is rejected by rejecting its second premiss.

Quine's argument for $V = L$ therefore rejects the same, second premiss of the indispensability argument as the naturalist fictionalist's argument, and also does so in exactly the same way, by claiming that the entities in question are not mentioned in our best scientific theory. Indeed, it is entirely fair to see Quine's attitude towards non-constructible sets as a mild naturalist fictionalism in its own right: the more sweeping naturalist fictionalist, like Field, simply extends Quine's considerations against non-constructible sets to include the constructible sets (and all entities which

[23]Ibid, my emphasis
[24]Cf Leng, 2005, 284n5 and Field, 1989, 45

can be built out of them, such as natural numbers and tensors). The difference between the two arguments is a difference of scope, not of spirit or in kind. If the naturalist fictionalist can make good on his programme of developing a nominalistic theory which generally surpasses the platonistic status quo, there appears to be no way a Quinean naturalist can retain her ontological commitment to even the more mundane mathematicals – indeed, she should enthusiastically embrace the opportunity to slim down her ontology.

4. Two problems for naturalistic fictionalism

But can the naturalist fictionalist make good on his programme? In this section, I give two arguments suggesting – but falling short of proving – he cannot.

4.1. **Balancing the epistemic virtues.** In the decades since 'Posits', Quineans, Kuhnians, and other philosophers of science – to say nothing of virtue epistemologists – have proposed several different lists of epistemic virtues.[25] For our purposes, it is most significant that none of these lists has consisted of a single element: each has contained a plurality of incommensurable and mutually irreducible epistemic goods. The 'goodness' of a theory is a multidimensional, not linear, quantity. In comparing a theory with a rival, it will do better with respect to some of these virtues, and worse with respect to others. To decide between the two theories, we must give weights to the several epistemic virtues. How much importance will we attach to ontological parsimony? Theoretical unity? Elegance of inference? And we must ask about tradeoffs: How much parsimony are we willing to sacrifice for the sake of fecundity? Can we sacrifice a certain minor amount of empirical adequacy if it means a great boost in theoretical unity? And so on.

Now, the naturalist cannot simply decide on how to weight the epistemic virtues, or defend a weighting by appealing to some *a priori* epistemology; he must defer to the way actual scientists weight the epistemic virtues. To do otherwise is to claim that the standards for our best scientific theory are not the ordinary standards of working scientists. In particular, working with the classical Quinean list to claim

[25]For a discussion of how radically these lists can diverge, and the consequences for philosophy of science as normative epistemology, see Longino, 1995.

the superiority of his nominalistic theory, he must claim that the gain in ontological parsimony greatly outweighs the combined loss of elegance of inference[26], theoretical unity[27], and fecundity[28] *in the eyes of ordinary working scientists*. While the naturalist fictionalist may be able to make this claim, I suspect he will have an extremely difficult time of it: scientists do, as a general rule, like to avoid postulating superfluous, unobservable entities, but would probably be extremely loathe to sacrifice their ability to actually engage in scientific research just to make philosophers happy. My suggestion, then, is that, contrary to the naturalist fictionalist's wishes, scientists generally place relatively little weight on ontological parsimony. It is therefore quite unlikely that the nominalistic theory can claim the title of best scientific theory.

My naturalist fictionalist has an 'easy' response here: show that ontological parsimony carries more weight among scientists than I claim. But this still means the naturalist fictionalist bears the burden of proof here. It is perhaps this burden which Field has in mind when he writes that defending fictionalism is a difficult undertaking.[29]

4.2. **Mathematical naturalism.** Maddy has offered extremely powerful naturalist critiques of both the Quinean indispensability argument and the argument for $V = L$[30]. Her claim, in essence, is that both arguments fail to be naturalistic if the proper naturalistic attitude towards mathematics is the same humility the naturalist shows towards physics and the other sciences. In light of the tight connection identified in §3, we shall restrict our attention to the argument for $V = L$.

Maddy's mathematical naturalism can be phrased as an addendum to the naturalist's account of 'best scientific theory'. As above, the Quinean naturalist takes the best scientific theory to be determined according to the ordinary standards of the working scientist. Maddy adds that the standards for the best scientific theory should also reflect the ordinary standards of the working mathematician. To do

[26]Cf the aardvark and bug example, §2
[27]Consider just the variety of elementary applications given a nearly uniform treatment by Fourier series in physics, as in Körner, 1989: the dynamics of heat, the analysis of sound waves, compass calibration, Brownian motion, the diameter of stars, code breaking and speed of practical computation calculations.
[28]Cf §4.2
[29]Cf n20, *supra*
[30]Maddy, 1997, III.6, 216-32, et al

otherwise is to either apologise for or critique mathematics from a perspective external to mathematics; and just as this would be naturalistically unacceptable with respect to physics, it is (or should be) naturalistically unacceptable with respect to mathematics.

With regard to $V = L$, Maddy claims that ontological parsimony is vehemently rejected by working mathematicians. Instead, they prefer a virtue of theories which she calls 'maximize' and which (very loosely) regards theory A more highly than theory B when A includes a wider variety of mathematicals for the mathematician to utilise in her constructions and investigate in their own right.[31] In the context of §4.1, this can be understood as the claim that, at least with respect to mathematical theories, fecundity (the strength and variety of research programmes which can be built on the theory) greatly outweighs ontological parsimony. The study of large cardinals, for example, and their relationship to the Continuum Hypothesis, model theory, the theory of subsets of the real numbers, &c, would be categorically ruled out – or, perhaps worse, denigrated to the pointless study of formal inference relations in a meaningless formal system[32] – by $V = L$.

Hence, Quine's argument for $V = L$ is not naturalist, at least according to Maddy's understanding of naturalism. As per the conclusion of §3, *a fortiori* neither is the naturalist fictionalist's argument.

The naturalist fictionalist – whether Quine or Field – might claim that this is begging the question against fictionalism. This is why I do not claim Maddy's arguments tell conclusively against the fictionalist. In response, we might challenge the fictionalist to explain why we should not automatically extend towards mathematicians the same deference we extended physicists, ie, why it is not begging the question against mathematics to presume that there is something suspicious about mathematicals. Here the fictionalist may have arguments[33], but they will not presuppose naturalism since the sense of either (1) or (1′), and thus must go beyond the scope of this paper.

[31] Ibid, 210-11
[32] Cf Quine, 1984, 788
[33] Cf Field, 1989, 67ff

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