THE HOLE ARGUMENT

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1. Statement of the Hole Argument

Let Σ be an open region of the n-dimensional spacetime manifold M such that Σ is contained entirely within the domain $U_\varphi = U$ of a chart φ of M. Denote the stress-energy tensor field by T, and the gravitation-metric tensor field by g.[1] Let $T_{\mu\nu} : \varphi(U) \to \mathbb{R}^{n\times n}$ be given by $d\varphi \circ \mathsf{T} \circ \varphi^{-1}$, and similarly $g_{\mu\nu} : \varphi(U) \to \mathbb{R}^{n\times n}$ be given by $d\varphi \circ \mathsf{g} \circ \varphi^{-1}$. Then the following diagram commutes[2]:

$$\begin{tikzcd}
T_{0,2}(U) \arrow[r, "d\varphi"] & \mathbb{R}^{n\times n} \\
U \arrow[u, "\mathsf{g}"] \arrow[r, "\varphi"] & \varphi(U) \arrow[u, "g_{\mu\nu}"']
\end{tikzcd}$$

And *mutatis mutandis* for T. Suppose that $T_{\mu\nu}$ is uniform over $\varphi(\Sigma)$.

Let K be the existing co-ordinate system (ie, including φ) on M considered as a topological manifold, and let K' be the co-ordinate system on M where φ is replaced with ψ, a chart on U which differs smoothly from φ on Σ. Then there is a map $g'_{\mu'\nu'} : \psi(U) \to \mathbb{R}^{n\times n}$ such that the following diagram commutes:

$$\begin{tikzcd}
T_{0,2}(U) \arrow[r, "d\psi"] & \mathbb{R}^{n\times n} \\
U \arrow[u, "\mathsf{g}"] \arrow[r, "\psi"] & \psi(U) \arrow[u, "g'_{\mu'\nu'}"']
\end{tikzcd}$$

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[1] Note that I am using $\mathsf{sans\ serif\ font}$ for tensor fields proper. Components of tensor fields with respect to a system of co-ordinates will be written in the usual indexed notation, viz, $g_{\mu\nu}$.

[2] $d\varphi$, the *differential* or *pushforward* of φ, is usually only officially defined on the tangent bundle TM; however, the definition on the tensor bundle $T_{0,2}(M)$ is entirely analogous. Cf Wald, 1984, 437ff

We will say that both $g_{\mu\nu}$ and $g'_{\mu'\nu'}$ are *parameterizations* of g with respect to the appropriate atlases: given the co-ordinates for a point p, they tell us the components of the tensor field g at p with respect to that system of co-ordinates. The relationship between $g_{\mu\nu}$ and $g'_{\mu'\nu'}$ is just one of a co-ordinate transform.

Without loss of generality, *as sets*, $\varphi(U) = \psi(U)$. Thus, while $\varphi(p) \neq \psi(p)$ in general, the image sets are, eg, both the unit sphere in $\mathbb{R}^n$. Hence, for a given $p \in U$, we can pass either the co-ordinates given by K or the co-ordinates given by K' to the numerical function g', ie, $g'_{\mu\nu}$ is well-defined. Define the tensor field g' such that the following diagram commutes:

$$\begin{tikzcd}
T_{0,2}(U) \arrow[r, "d\varphi"] & \mathbb{R}^{n\times n} \\
U \arrow[u, "\mathsf{g}'"] \arrow[r, "\varphi"] & \varphi(U) \arrow[u, "g'_{\mu\nu}"']
\end{tikzcd}$$

This tensor field is not equal to g. For, if $\mathsf{g} = \mathsf{g}'$, then the following diagram would commute:

$$\begin{tikzcd}
\mathbb{R}^{n\times n} & T_{0,2}(U) \arrow[l, "d\varphi"'] \arrow[r, "d\varphi"] & \mathbb{R}^{n\times n} \\
\varphi(U) \arrow[u, "g_{\mu\nu}"] & U \arrow[l, "\varphi"'] \arrow[r, "\varphi"] \arrow[u, "\mathsf{g}=\mathsf{g}'"] & \varphi(U) \arrow[u, "g'_{\mu\nu}"']
\end{tikzcd}$$

However, this diagram does not commute. For example, in a dimension 2 spacetime, if $\varphi(p) = (1,1)$ for some $p \in \Sigma$ and

$$g_{\mu\nu}(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \tag{1}$$

$$g'_{\mu\nu}(1,1) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{2}$$

Define $T'_{\mu\nu}$ and T' *mutatis mutandis.* In this case, though, $\mathsf{T}' = \mathsf{T}$ and $T'_{\mu\nu} = T_{\mu\nu}$ throughout their respective domains.

It is important to note that the construction of g' is more complicated than just a point or co-ordinate transform. In the final step of the construction, *both* the co-ordinate system and tensor field are 'dragged' relative to the points of M.

Suppose that our field equations are *not* written in a co-ordinate-free form. For example, as an equation in terms of the matrix-valued functions $g_{\mu\nu}$ and $T_{\mu\nu}$. Then $\langle M, K, T_{\mu\nu}, g_{\mu\nu}\rangle$ is a *parameterized solution* to these field equations, by hypothesis. Suppose that our field equations are, in addition, generally covariant. In this non-co-ordinate-free setting, that means the algebraic relation expressed by the field equation between $T_{\mu\nu}$ and $g_{\mu\nu}$ also holds between $T'_{\mu\nu} = T_{\mu\nu}$ and $g'_{\mu\nu}$. Hence $\left\langle M, K, T_{\mu\nu}, g'_{\mu\nu}\right\rangle$ is a distinct parameterized solution to these same field equations, ie, the matter-energy field does not uniquely determine the gravitational field.

2. Diffeomorphic equivalence

By far, the most common proposed solution to the Hole Argument is to appeal to *diffeomorphic equivalence*. This notion is usually only defined for non-parameterized solutions (that is, solutions to co-ordinate-free field equations)[3]. I extend this definition to parameterized solutions.

> **Diffeomorphic Equivalence:** Let $\mathfrak{M}_1 = \langle M, K, T_{\mu\nu}, g_{\mu\nu}\rangle$ and $\mathfrak{M}_2 = \left\langle M, K', T'_{\mu'\nu'}, g'_{\mu'\nu'}\right\rangle$ be mathematically distinct parameterized solutions. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to be *equivalent up to diffeomorphism* or *diffeomorphically equivalent* if and only if there is a diffeomorphism $f : M \to M$ such that $\mathfrak{M}_2 = \langle M, f * K, f * T_{\mu\nu}, f * g_{\mu\nu}\rangle$. Any two solutions which are equivalent up to diffeomorphism are physically identical.

For example, Wald appeals to the non-parameterized notion when he writes that 'Any physically meaningful statement about [the first solution] $(M, T^{(i)})$ will hold with equal validity for [the second solution] $(N, \phi * T^{(i)})$'[4]. Earman[5] cites Hawking and Ellis appealing to diffeomorphic equivalence to solve Hole Argument-type problems. Norton writes that 'Whatever indeterminism is revealed in the hole argument is a purely mathematical freedom akin to a gauge freedom and offers no obstacle

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[3] Cf Earman, 1989, 185-6

[4] Wald, 1984, 438

[5] Op cit

to the physical interest of a generally covariant theory'[6]. And Howard and Norton write that 'Hertz's proposal …would have provided a serviceable escape from the hole argument', a proposal very closely related to Hilbert's proposal that 'something should be regarded as physically meaningful only if it is invariant with respect to arbitrary transformations of the coordinate system'[7].

However, this approach will not work for the Hole Argument as I have presented it. There is no diffeomorphism f such that $\langle M, K, T'_{\mu\nu}, g'_{\mu\nu} \rangle = \langle M, f * K, f * T, f * G \rangle$, since the only diffeomorphism that preserves K is the identity 1_M. While diffeomorphic equivalence does solve the co-ordinate free version of the problem – the two solutions, $\langle M, \mathsf{T}, \mathsf{g} \rangle$ and $\langle M, \mathsf{T}, \mathsf{g}' \rangle$, are equivalent by way of the diffeomorphism $\psi^{-1} \circ \varphi$ since $\mathsf{T} = \mathsf{T}'$ – the co-ordinate system K provides extra structure that dramatically restricts the class of equivalence-making diffeomorphisms, and hence dramatically narrows the equivalence classes.

At this point, the reader may object that my presentation of the Hole Argument is just an incidental curiosity. One result to be taken away from the last paragraph is that the Hole Argument only presents a problem which cannot be solved by a standard equivalence relation if we insist on not moving to co-ordinate-free versions of the field equations; but standard practice today is to use co-ordinate-free versions, therefore &c. In the remainder of this paper, I will defend the following two claims:

(1) The Hole Argument, as I have presented it, is a more faithful presentation of Einstein's version of the argument than the co-ordinate-free version. My presentation is therefore of historical significance.

(2) I distinguish between 'trivial' and 'non-trivial' understandings of general covariance. Non-trivial general covariance is susceptible to the Hole Argument as I have presented it. Hence my presentation is of philosophical significance if only as a *reductio* of non-trivial general covariance. This may have implications for the debate over the physical significance of general covariance, though I will not discuss that issue in any depth in this paper.

First, however, I take a moment to explain how to actually solve the Hole Argument.

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[6] Norton, 1993, 805

[7] Howard & Norton, 1993, 50-1

3. Reparameterization equivalence

The solution begins by considering the so-called Point-Coincidence Argument. This Argument is given by Einstein as follows.

> Reality is nothing but the totality of space-time point coincidences. If, for example, physical happenings could be built up out of the motion of material points alone, then the meetings of the points, i.e., the points of intersection of their world lines, would be the only reality, i.e., in principle observable. Naturally these points of intersection remain unchanged in all transformations (and no new ones are added) only if certain uniqueness conditions are preserved. Therefore it is most natural to require of laws that they determine no more than the totality of timespace coincidences. Following what has been said before, this is already achieved with generally covariant equations.[8]

I first read this Argument as articulating the following principle:

> **Individuation:** A spacetime point-event p is physically individuated by the confluence of physical properties located at p.

Here a 'physical property' includes all the values (in some co-ordinate system) of physically significant object-fields and determinants of determinables of ponderable matter (eg, the mass of a point-mass whose worldline passes through p) detailed in the theory. 'Physical individuation' should be understood along the lines of 'individuated by the theory', or even merely epistemologically; it is not, in particular, a metaphysical thesis of the constitution of the numerical identity of point-events.

Even with these caveats, however, Individuation is too strong. Suppose p and p' are topologically distinct point-events in some solution $\langle M, K, T_{\mu\nu}, g_{\mu\nu} \rangle$ of parameterized GTR, and suppose in addition that

$$(T_{\mu\nu} \circ \varphi)(p) = (T_{\mu\nu} \circ \psi)(p'),\ (g_{\mu\nu} \circ \varphi)(p) = (g_{\mu\nu} \circ \psi)(p')$$

for every pair of charts φ, ψ in K such that $p \in U_\varphi$ and $p' \in U_\psi$. Then, according to Individuation, p and p' are physically identical.

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[8] Einstein, quoted in Norton, 1984, 290-1

The problem here is that Individuation is supposed to be used to identify point-events *across* different solutions, not *within* one solution. It is to be used to define an equivalence relation between solution such that the new solution produced by the Hole Argument is equivalent to the old, not actually identify individual point-events. Hence, the thrust of the Point-Coincidence Argument is better captured by the following:

> **Reparameterization Equivalence:** Let $\mathfrak{M}_1 = \langle M, K, T_{\mu\nu}, g_{\mu\nu} \rangle$ and $\mathfrak{M}_2 = \left\langle M, K, T'_{\mu\nu}, g'_{\mu\nu} \right\rangle$ be two solutions to the parameterized field equations. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to be *reparameterization equivalent*, or $\mathfrak{M}_2$ is said to be a *reparameterization* of $\mathfrak{M}_1$ if and only if there is an atlas K' compatible with the atlas K such that, for every $\varphi \in K$, $\psi \in K'$, and $U = U_\varphi \cap U_\psi$ the following diagram commutes[9]:
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> $$\begin{tikzcd}
\mathbb{R}^{n\times n} & T_{0,2}(U) \arrow[l, "d\varphi"'] \arrow[r, "d\psi"] & \mathbb{R}^{n\times n} \\
\varphi(U) \arrow[u, "g_{\mu\nu}"] & U \arrow[l, "\varphi"'] \arrow[r, "\psi"] \arrow[u, "\mathsf{g}"'] & \psi(U) \arrow[u, "g'_{\mu'\nu'}"']
\end{tikzcd}$$
>
> And *mutatis mutandis* for $T_{\mu\nu}$ and $T'_{\mu\nu}$. Then any two reparameterization equivalent solutions are physically equivalent.

Just as diffeomorphic equivalence was, essentially, the stipulation that solutions related by the construction in Earman's version of the Hole Argument are physically equivalent, reparameterization equivalence is the stipulation that solutions related by the construction in my version of the Hole Argument are physically equivalent. Note that reparameterization equivalence is *not* just gauge freedom or Hilbert's invariance with respect to co-ordinate transformations. Those two terms express the notion that one and the same physical situation can be redescribed with respect to different co-ordinate systems. Reparameterization equivalence asserts that, in addition, one and the same physical situation can be redescribed with respect to

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[9] The reference to the tensor field g is included simply to make clear the relationship between this diagram and the diagrams in the presentation of the Hole Argument. A simpler diagram where reference is made to neither U nor its tensor bundle could also work for the purposes of giving this definition.

one and the same co-ordinate system, by way of a second co-ordinate system and general covariance.

Hence, to use some language developed in the next section, reparameterization equivalence, in line with the Point-Coincidence Argument, asserts that the atlas K is not an individuating structure, ie, the co-ordinates K assigns to a point p are not physically significant.

4. Einstein's Hole Argument

According to Norton[10], Einstein's fourth (1914) version of the Hole Argument is the most complete, and the only statement in which it is clear that Einstein is making a crucial distinction between, in my terminology, the re-parameterization $g'_{\mu'\nu'}$ of the original metric field g (a simple co-ordinate transform) and the new metric field g' (a more complex construction involving both a point and co-ordinate transform). As presented by Norton, it runs as follows:

> We consider a finite region of the continuum Σ, in which no material process takes place. Physical happenings in Σ are then fully determined, if the quantities $g_{\mu\nu}$ are given as functions of the x_ν in relation to the coordinate system K used for description. The totality of these functions will be symbolically denoted by $G(x)$.
>
> Let a new coordinate system K' be introduced, which coincides with K outside Σ, but deviates from it inside Σ in such a way that the $g'_{\mu\nu}$ related to the K' are continuous everywhere like the $g_{\mu\nu}$ (together with their derivatives). We denote the totality of the $g'_{\mu\nu}$ symbolically with $G'(x')$. $G'(x')$ and $G(x)$ describe the same gravitational field. In the functions $g'_{\mu\nu}$ we replace the coordinates x'_ν with the coordinates x_ν, i.e., we form $G'(x)$. Then, likewise, $G'(x)$ describes a gravitational field with respect to K, which however does not correspond with the real (or originally given) gravitational field.
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> We now assume that the differential equations of the gravitational field are generally covariant. Then they are satisfied by $G'(x')$ (relative

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[10] Norton, 1984, 286-7

> to K') if they are satisfied by $G(x)$ relative to K. Then they are also satisfied by $G'(x)$ relative to K. Then relative to K there exist the solutions $G(x)$ and $G'(x)$, which are different from one another, in spite of the fact that both solutions coincide in the boundary region, i.e., *happenings in the gravitational field cannot be uniquely determined by generally covariant differential equations for the gravitational field.*[11]

There are two critical aspects of this argument that distinguish it from the co-ordinate-free versions presented by, eg, Earman[12]. First, the use of the functions $G(x), G'(x')$, and $G'(x)$. Properly speaking, these are *not* tensor fields on Σ. Rather, they are parameterizations of tensor fields with respect to the co-ordinate systems K, K', and K, respectively. In my terminology, they are the functions $g_{\mu\nu}, g'_{\mu'\nu'}$, and $g'_{\mu\nu}$.

For example, when Einstein writes that the assumed generally covariant field equations are 'satisfied by $G'(x')$ (relative to K') if they are satisfied by $G(x)$ relative to K', he must be taking the field equations to be equations between the matrix-valued functions $g_{\mu\nu}$ and $T_{\mu\nu}$ (in my notation). That is, the field equations used here are *not* co-ordinate free. Furthermore, the construction of $g'_{\mu\nu}$ makes essential use of the change of co-ordinates from K to K' and back again, and not just in a passive way: the active point transform (the 'dragging' of the metric tensor field) is built on the second co-ordinate transform (from K' back to K). It is this step which Norton identifies as the crucial distinction between this version of the Argument and previous versions, and which means Einstein's construction is not just a simple confusion of active and passive transforms.[13] By contrast, not only does Earman's version not make use of passive co-ordinate transforms, his setting is entirely co-ordinate-free – no systems of co-ordinates are even mentioned.

Second, this essential use of the co-ordinate system means there is no proper diffeomorphism in Einstein's construction, or at least no non-trivial diffeomorphism.

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[11] Einstein, quoted in Norton, 1984, 288, emphasis in Norton

[12] Earman, 1989, 175-7, which, in what follows, I will take to be the canonical modern presentation of the Hole Argument

[13] Norton, 1984, 287-8; cf Janssen, 2006, 819-23

Replacing φ with ψ, ie, replacing K with K', does not change the differential structure on the topological manifold M at all – recall that the differential structure is a maximal atlas or an equivalence class of compatible atlases which already includes both K and K' – and hence does nothing at all to the vector and tensor fields built out of that differential structure, either. The introduction of a co-ordinate system picks out a particular representative of the differential structure. That is, the choice of K or K' adds additional structure to the differential manifold. In Stachel's terminology, K serves as an *individuating structure* for the spacetime points, *prior to* or *independently of* the metric field.[14] Einstein's construction moves the fields around while keeping the points of the spacetime manifold in place using the co-ordinate system.

This is why equivalence up to diffeomorphism cannot solve the Hole Argument in my presentation: an arbitrary diffeomorphism $M \to M$ preserves the topology and differential structure, but does not preserve the choice of atlas K unless it is the identity map. That is, the only f such that the following diagram commutes is 1_U:

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[d, "f"'] & \varphi(U) \arrow[d, "\psi\circ\varphi^{-1}"] \\
U \arrow[r, "\psi"] & \psi(U)
\end{tikzcd}$$

Suppose $p \in \Sigma$ is such that $f(p) = p'$ where $p' \neq p$. Then

$$(\psi \circ \varphi^{-1} \circ \varphi)(p) = \psi(p) \tag{3}$$

while

$$(\psi \circ f)(p) = \psi(p'). \tag{4}$$

And since $p \neq p'$ and ψ is 1-1 on Σ, $\psi(p) \neq \psi(p')$. In particular, the 'co-ordinate change diffeomorphism' given (locally) by $\psi^{-1} \circ \varphi$, which takes p to the point p' whose co-ordinates in K' are the same as co-ordinates of p in K (that is, the point p' such that $\psi(p') = \varphi(p)$), does not make this diagram commute.

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[14] Stachel, 1989, 75

On the other hand, as in Earman, moving to a co-ordinate-free setting means we no longer need to preserve the choice of atlas K – it no longer serves as an individuating structure independent of the tensor fields – and hence, in this setting, we can use a diffeomorphism to identify the two solutions.

Now, observe that my presentation of the Hole Argument preserves precisely these two features. I explicitly use parameterized field equations, and a pair of co-ordinate transforms instead of a diffeomorphism. I do not see any element of Einstein's version of the Argument which does not appear in my version, though I have taken advantage of the more precise notation of contemporary differential topology to more accurately state the Argument. I conclude that my presentation is more faithful to Einstein's than Earman's co-ordinate-free version precisely because my version is not co-ordinate-free.

This leads to an important question: Why has no-one else recognised this difference between the two versions? I would like to identify three factors that contributed to this failure.

First, Einstein himself did not consistently present the Hole Argument with the crucial final step (the move from $g'_{\mu'\nu'}$ to $g'_{\mu\nu}$ and thus to the construction of the new tensor field g'). Norton identifies four versions of the Argument, only the last of which includes the final step.[15] Without this step, we have just the two parameterizations $g_{\mu\nu}$ and $g'_{\mu'\nu'}$ of one and the same field g, which are easily identified using diffeomorphism equivalence.

Second, Ryckman has argued, following the work by Norton and Stachel, that the Point-Coincident Argument was appropriated and deeply misread by the logical empiricists as verificationist in spirit.[16] It is not until Stachel's recovery of both Arguments in the 1980s that this verificationist reading is soundly rejected.[17] However, the tools of the study of differentiable manifolds in 1980 were significantly advanced compared to 1914. In particular, mathematical physicists now had the resources to work in co-ordinate-free settings. Hence, for convenience, Stachel revives the

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[15] Norton, 1984, 286ff

[16] Ryckman, 1992, 475

[17] Ryckman, 1992, 494

Hole Argument in a form that more closely resembles Earman's presentation than Einstein's.[18]

Stachel's solution is more subtle than equivalence up to diffeomorphism, however. He considers the possibility of fields or other 'individuating structures' which can serve to individuate points but are *not* 'dragged along' when the metric field g is replaced with g'. 'If such fields exist, then all of the solutions to the generally covariant field equations generated from the initial one by dragging must be regarded as physically distinct, and . . . the "hole" argument is valid.'[19] The solution, as in the Point-Coincidence argument, is that such fields do not exist, or at least have no physical significance. But Stachel simply takes it for granted that a co-ordinate system has no physical significance. In the co-ordinate-free context, of course, this is entirely viable; but in a non-co-ordinate-free context, the choice of co-ordinate system is logically prior to the statement of the field equations, and hence might very well be of physical significance.[20] Developing this point is the topic of the next section. For now, it suffices to note that, when the true thrust of the Hole Argument was recovered in the 1980s, the natural presentation was a co-ordinate-free version which is solved using diffeomorphic equivalence.

Third, as mentioned above, Hilbert had an argument that resembles the Hole Argument up to the last step. As glossed by Howard and Norton:

> [Hilbert's] theory has fourteen independent variables . . . but the gravitational field equations and Maxwell's equations provide only ten independent field equations. Hilbert illustrates this underdetermination with a pair of solutions, the first of which represents an electron at rest throughout all time . . . . [while] the second solution is obtained by a

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[18] Stachel, 1989, 74ff

[19] Op cit, 79

[20] While this proposal does and should strike us as absurd today, in any non-generally covariant presentation of a theory of spacetime, the choice of co-ordinate system will be restricted, and hence could be said to have physical significance. In classical Newtonian spacetime, for example, there is an observer-independent fact of the matter whether two point-events are simultaneous. In this spacetime, point-events are individuated by an arbitrary representative of the equivalence class of inertial reference frames prior to the introduction of matter fields, and therefore the equivalence class is an individuating structure.

> coordinate transformation that is the identity for the time coordinate $x_4 \leq 0$, but comes to differ for $x_4 > 0$.[21]

As they, and Hilbert, point out, this underdetermination is easily solved by appeal to equivalence up to diffeomorphism: in the change of co-ordinates, we have simply redescribed one and the same physical situation.

In short, for several decades, the Hole Argument is presented as an issue either easily solved by diffeomorphic equivalence or tied into a crude verificationism. By the time its original force is recovered, the general prevalence of co-ordinate-free presentations of the field equations means the non-co-ordinate-free presentation is neglected.

5. Two types of general covariance

Earman characterises general covariance as follows: a system of laws is called generally covariant if and only if it consists of 'partial differential equations written in terms of geometric-object fields . . . [that] have a form that is the same in every co-ordinate system'[22]. Paradigmatic geometric-object fields are (smooth) vector fields and tensor fields.[23]

This characterisation, which I take to be perfectly standard, fails to discriminate between what I will call *trivial* and *non-trivial* ways a theory can be generally covariant.

Consider the Einstein field equations as sometimes presented today. In my notation, and for some choice of units and some formula Φ in the language of the tensor calculus, these take the form

$$\forall p \in M,\ \ \Phi(\mathsf{g}(p), \mathsf{T}(p)) \tag{5}$$

That is, they take the form of an equation between two tensors, and in particular are co-ordinate-free. Hence, these equations are (properly, 'this equation is') *trivially generally covariant*: they have a form which is the same in (with respect to) every co-ordinate system because it (the equational form) makes absolutely no

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[21] Op cit, 49

[22] Earman, 1989, 47

[23] Op cit, 159

reference whatsoever to any co-ordinate system. One can choose any arbitrary atlas K, or indeed not make any choice of atlas at all. Hence, trivial general covariance is the requirement that the laws of the theory be stated without reference to any co-ordinate system (in an essential way[24]). At least as a first pass, trivial general covariance is the same as Norton's 'automatic general covariance'[25] and Wald's definition of general covariance as the requirement that 'the metric of space is the only quantity pertaining to space that can appear in the laws of physics'[26]. In particular, no co-ordinate system can serve as a pre-physical individuating structure.

Now consider the parameterized field equations as they would appear in a particular instance of the Hole Argument as I have presented it. For some choice of units and some formula Ψ in the language of real matrix algebra, these take the form

$$\forall \phi \in K \ \forall \mathbf{x} \in \phi(U_\phi), \ \ \Psi(g_{\mu\nu}(\mathbf{x}), T_{\mu\nu}(\mathbf{x})) \tag{6}$$

This is an equation between two matrices, and whether it holds for a given $g_{\mu\nu}, T_{\mu\nu}$, and Ψ depends, *prima facie*, on the choice of atlas K. This equation is *non-trivially generally covariant*: that the holding of this equation does not actually depend on K must be proved. We can also understand non-trivial general covariance as saying that any parameterization of the geometric-object fields – *not* the fields themselves – must satisfy a particular equality. That is, the laws of the theory make reference to some co-ordinate system (in an essential way), and hold whatever co-ordinate system is actually chosen.

Next suppose that generally covariant *simpliciter* means non-trivially generally covariant. This would be one way to attempt to claim physical significance for general covariance, for example. Then the field equations in their standard, co-ordinate-free form are *not* generally covariant, and neither is any co-ordinate-free presentation of any theory of kinematics or gravitation. On this understanding of general covariance, the laws of a theory must be presented in parameterized form, and then shown to hold with respect to any particular parameterization.

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[24] This clause is to prevent trivial and silly additions such as 'Pick an atlas K such that K is an atlas'.

[25] Norton, 1993, 821ff

[26] Wald, 1984, 57

And finally note that trivial general covariance is the understanding of general covariance employed in Earman's presentation of the Hole Argument – namely, that the solution does not need to include reference to a choice of atlas K – while mine employs non-trivial general covariance. In particular, in the final change of coordinates, where the new tensor field $g'_{\mu\nu}$ is constructed, it is precisely the formal algebraic stability of the field equations to which the Argument appeals. Hence, if general covariance *simpliciter* means non-trivial general covariance, Earman's presentation does not appeal to general covariance *simpliciter*.

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