An upper bound for the number of independent sets in regular graphs

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Abstract

Write $\mathcal{I}(G)$ for the set of independent sets of a graph G and $i(G)$ for $|\mathcal{I}(G)|$. It has been conjectured (by Alon and Kahn) that for an N-vertex, d-regular graph G,

$$i(G) \leq \left(2^{d+1} - 1\right)^{N/2d}.$$

If true, this bound would be tight, being achieved by the disjoint union of $N/2d$ copies of $K_{d,d}$. Kahn established the bound for bipartite G, and later gave an argument that established

$$i(G) \leq 2^{\frac{N}{2}\left(1+\frac{2}{d}\right)}$$

for G not necessarily bipartite. In this note, we improve this to

$$i(G) \leq 2^{\frac{N}{2}\left(1+\frac{1+o(1)}{d}\right)}$$

where $o(1) \to 0$ as $d \to \infty$, which matches the conjectured upper bound in the first two terms of the exponent.

We obtain this bound as a corollary of a new upper bound on the independent set polynomial $P(\lambda, G) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$ of an N-vertex, d-regular graph G, namely

$$P(\lambda, G) \leq (1+\lambda)^{\frac{N}{2}} 2^{\frac{N(1+o(1))}{2d}}$$

valid for all $\lambda > 0$. This also allows us to improve the bounds obtained recently by Carroll, Galvin and Tetali on the number of independent sets of a fixed size in a regular graph.

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1 Introduction

For a (simple, finite, undirected) graph G write $\mathcal{I}(G)$ for the set of independent sets of G (sets of vertices no two of which are adjacent) and $i(G)$ for $|\mathcal{I}(G)|$. How large can $i(G)$ be? For the class of N-vertex, d-regular graphs, this question has received some attention, with a succession of bounds having appeared going back to the early 1990's.

A trivial upper bound is $i(G) \leq 2^N$. In [1], Alekseev gave the first non-trivial bound, establishing $i(G) \leq 3^{\frac{N}{2}}$. In fact he showed that for any graph G on N vertices (not necessarily regular)

$$i(G) \leq \left(1 + \frac{N}{\alpha}\right)^{\alpha} \tag{1}$$

where $\alpha = \alpha(G)$ is the size of the largest independent set in G; note that for N-vertex, d-regular G, $\alpha(G) \leq \frac{N}{2}$. Around the same time, in the process of resolving a question of Erdős and Cameron on sum-free sets in Abelian groups, Alon [3] substantially improved this to

$$i(G) \leq \exp_2\left\{\frac{N}{2}\left(1 + \frac{C}{d^{1/10}}\right)\right\} \tag{2}$$

for some constant $C > 0$. This bound is best possible in the leading term of the exponent: the graph $\frac{N}{2d}K_{d,d}$ consisting of a disjoint union of $N/2d$ copies of $K_{d,d}$ satisfies

$$i\left(\frac{N}{2d}K_{d,d}\right) = \left(2^{d+1} - 1\right)^{\frac{N}{2d}} = \exp_2\left\{\frac{N}{2}\left(1 + \frac{1}{d} - \frac{1 + o(1)}{(2\ln 2)d2^d}\right)\right\}$$

where $o(1) \to 0$ as $d \to \infty$. Alon speculated that perhaps among all N-vertex, d-regular graphs, $\frac{N}{2d}K_{d,d}$ is the one that admits the greatest number of independent sets.

Conjecture 1.1 *For any N-vertex, d-regular graph G,*

$$i(G) \leq \left(2^{d+1} - 1\right)^{\frac{N}{2d}}.$$

Kahn [7] used entropy methods to prove the above bound for N-vertex, d-regular bipartite graphs, and in the same paper formally conjectured that the bound should hold for all graphs.

Further progress was made by Sapozhenko [10], who used a very simple counting argument to improve (2) to

$$i(G) \leq \exp_2\left\{\frac{N}{2}\left(1 + C\sqrt{\frac{\log d}{d}}\right)\right\}$$

for some constant $C > 0$. (In this note "$\log$" will always indicate the base 2 logarithm).

The next substantial improvement was made by Kahn (personal communication to the author; the proof appears in [8], where it is generalized to the context of graph homomorphisms), who obtained

$$i(G) \leq \exp_2\left\{\frac{N}{2}\left(1 + \frac{2}{d}\right)\right\}. \tag{3}$$

The aim of this note is to improve (3) to the following.

Theorem 1.2 *There is a constant $C > 0$ such that for any d-regular, N-vertex graph G,*

$$i(G) \leq \exp_2\left\{\frac{N}{2}\left(1+\frac{1}{d}+\frac{C}{d}\sqrt{\frac{\log d}{d}}\right)\right\}.$$

This matches the first two terms in the exponent of $i\left(\frac{N}{2d}K_{d,d}\right)$. The proof of Theorem 1.2 combines the idea used to prove (3) with a recent theorem of Sapozhenko bounding the number of independent sets in a regular graph in terms of the size of the largest independent set (see Lemma 2.3). The basic idea is to treat two cases. If G has no large independent sets, then Sapozhenko's result implies that it has few independent sets. On the other hand if G has a large independent set then it is close to bipartite (the case for which Conjecture 1.1 is resolved) and the method used to prove (3) can be modified to exploit this fact and obtain a bound closer to that of Conjecture 1.1.

What we actually prove is a weighted generalization of Theorem 1.2. The independent set (or stable set) polynomial of G (first introduced explicitly by Gutman and Harary [6]) is defined as

$$P(\lambda, G) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$$

This is also referred to as the partition function of the independent set (or hard-core) model on G with activity λ. In [5] the analog of Conjecture 1.1 was obtained for N-vertex, d-regular bipartite G:

$$P(\lambda, G) \leq (2(1+\lambda)^d - 1)^{\frac{N}{2d}} \quad \left(= P\left(\lambda, K_{d,d}\right)^{\frac{N}{2d}}\right)$$

for all $\lambda > 0$ (the case $\lambda \geq 1$ was already dealt with in [7]), and it was conjectured that this bound should hold for non-bipartite G also. In [4] the analog of (3) was obtained for N-vertex, d-regular G:

$$P(\lambda, G) \leq (1+\lambda)^{\frac{N}{2}} 2^{\frac{N}{d}}. \tag{4}$$

By employing a weighted generalization of (1) (see Lemma 2.2) we improve (4) to the following.

Theorem 1.3 *For all $\lambda > 0$ there is a constant $C_\lambda > 0$ such that for d-regular, N-vertex G,*

$$P(\lambda, G) \leq (1+\lambda)^{\frac{N}{2}} \exp_2\left\{\frac{N}{2d}\left(1 + C_\lambda\sqrt{\frac{\log d}{d}}\right)\right\}.$$

Note that this reduces to Theorem 1.2 when $\lambda = 1$.

Theorem 1.3 has consequences for the number of independent sets of a fixed size in a regular graph. With regards to this, Kahn [7] made the following conjecture. Here $i_t(G)$ is the number of independent sets in G of size t.

Conjecture 1.4 *If G is an N-vertex, d-regular graph with $2d|N$, then for each $0 \leq t \leq N$,*

$$i_t(G) \leq i_t\left(\frac{N}{2d}K_{d,d}\right).$$

In [4], asymptotic evidence is provided for this conjecture in the sense that if N, d and t are sequences satisfying $t = \alpha N/2$ for some fixed $\alpha \in (0,1)$ and G is a sequence of N-vertex, d-regular graphs, then

$$i_t(G) \leq \begin{cases} \exp_2\left\{\frac{N}{2}\left(H(\alpha) + \frac{2}{d}\right)\right\} & \text{in general and} \\ \exp_2\left\{\frac{N}{2}\left(H(\alpha) + \frac{1}{d}\right)\right\} & \text{if } G \text{ is bipartite,} \end{cases} \tag{5}$$

where $H(\cdot)$ is the binary entropy function. On the other hand, if $N = \omega(d \log d)$ and $d = \omega(1)$ then

$$i_t\left(\frac{N}{2d}K_{d,d}\right) \geq \exp_2\left\{\frac{N}{2}\left(H(\alpha) + \frac{1-o(1)}{d}\right)\right\}$$

(all as $d \to \infty$). Using Theorem 1.3 in place of (4) in the derivation of the first bound in (5) we get the immediate improvement that there is a constant $c_\alpha > 0$ such that for all N-vertex, d-regular G

$$i_t(G) \leq \exp_2\left\{\frac{N}{2}\left(H(\alpha) + \frac{1}{d} + \frac{c_\alpha}{d}\sqrt{\frac{\log d}{d}}\right)\right\},$$

so that the upper bound for general G matches the conjectured bound in the first two terms of the exponent (in the range $N = \omega(d \log d)$, $d = \omega(1)$ and $t = \alpha N/2$).

Section 2 gives the three lemmas that we need for the proof of Theorem 1.3, while the proof of the theorem is given in Section 3.

2 Tools

We begin by recalling a result from [4] which is a slight refinement of (4) (see the derivation of (10) in that reference). For a total order $\prec$ on $V(G)$ and for each $v \in V(G)$ write $P_\prec(v)$ for $\{w \in V(G) : \{w,v\} \in E(G),\ w \prec v\}$ and $p_\prec(v)$ for $|P_\prec(v)|$. Note that $\sum_{v \in V(G)} p_\prec(v) = |E(G)|$ ($= Nd/2$ when G is N-vertex and d-regular).

Lemma 2.1 *For any d-regular, N-vertex graph G and any total order $\prec$ on $V(G)$,*

$$P(\lambda, G) \leq \prod_{v \in V(G)} \left(2(1+\lambda)^{p_\prec(v)} - 1\right)^{\frac{1}{d}}.$$

Next, we give a weighted generalization of (1).

Lemma 2.2 *For any N-vertex graph G (not necessarily regular) with $\alpha(G) = \alpha$, and any $\lambda > 0$,*

$$P(\lambda, G) \leq \left(1 + \frac{\lambda N}{\alpha}\right)^\alpha$$

with equality if and only if G is the disjoint union of complete graphs all of the same order.

Proof: We follow closely the proof of (1) that appears in [2], making along the way the changes needed to introduce λ.

We first observe that it is enough to prove the bound for connected G. Indeed, if G has components $C_1, \ldots, C_n$ with $|C_i| = N_i$ and $\alpha(G[C_i]) = \alpha_i$ then $\sum N_i = N$ and $\sum \alpha_i = \alpha$. Using Jensen's inequality for (6) we have

$$\begin{aligned} P(\lambda, G) &= \prod_{i=1}^{n} P(\lambda, G[C_i]) \\ &\leq \prod_{i=1}^{n} \left(1 + \frac{\lambda N_i}{\alpha_i}\right)^{\alpha_i} \\ &\leq \left(\frac{\sum_{i=1}^{n} \alpha_i \left(1 + \frac{\lambda N_i}{\alpha_i}\right)}{\alpha}\right)^{\alpha} \qquad (6) \\ &= \left(1 + \frac{\lambda N}{\alpha}\right)^{\alpha}, \end{aligned}$$

with equality in (6) if and only if all the N_i are equal (i.e., all components of G have the same order).

We prove the bound for connected G by induction on the number of vertices, with the single-vertex case trivial. For connected G with more than one vertex, let v be a vertex of maximum degree Δ. We use the recurrence

$$P(\lambda, G) = P(\lambda, G - v) + \lambda P(\lambda, G - v - N(v))$$

(where $N(v)$ is the set of vertices in G adjacent to v), which follows from the fact that there is a bijection from independent sets of size t in G which do not contain v to independent sets of size t in $G - v$, and a bijection from independent sets of size t in G which do contain v to independent sets of size $t-1$ in $G - v - N(v)$. Since $\alpha(G - v) \leq \alpha$ and $\alpha(G - v - N(v)) \leq \alpha - 1$ we have by induction and the fact that $\left(1 + \frac{a}{x}\right)^x$ is increasing in $x > 0$ for all $a > 0$

$$P(\lambda, G) \leq \left(1 + \frac{\lambda(N-1)}{\alpha}\right)^{\alpha} + \lambda\left(1 + \frac{\lambda(N - \Delta - 1)}{\alpha - 1}\right)^{\alpha - 1}. \qquad (7)$$

We upper bound the right-hand side of (7) by observing that $\Delta \geq \frac{N-1}{\alpha}$. For G complete or an odd cycle, this is immediate, and for all other connected G the stronger bound $\Delta \geq \frac{N}{\alpha}$ follows from Brooks' theorem. Inserting into (7) we obtain

$$P(\lambda, G) \leq \left(1 + \frac{\lambda(N-1)}{\alpha}\right)^{\alpha} + \lambda\left(1 + \frac{\lambda(N-1)}{\alpha}\right)^{\alpha - 1}$$

and so

$$\begin{aligned} \frac{P(\lambda, G) - \left(1 + \frac{\lambda N}{\alpha}\right)^{\alpha}}{\left(1 + \frac{\lambda(N-1)}{\alpha}\right)^{\alpha}} &\leq 1 + \frac{\lambda\alpha}{\alpha + \lambda(N-1)} - \left(1 + \frac{\lambda}{\alpha + \lambda(N-1)}\right)^{\alpha} \\ &\leq 0 \end{aligned}$$

with equality if and only if $\alpha = 1$ (i.e., G is a complete graph). $\square$

Finally, we give a weighted variant of a recent result of Sapozhenko [9] bounding the number of independent sets in a regular graph in terms of $\alpha(G)$.

Lemma 2.3 *There is a constant $c > 0$ such that for any d-regular, N-vertex graph G with $d \geq 2$ and $\alpha(G) = \alpha$ and any $\lambda > 0$,*

$$P(\lambda, G) \leq \left(1 + \frac{\lambda N}{2\alpha}\right)^{\alpha} \exp_2\left\{cN\sqrt{\frac{\log d}{d}}\right\}.$$

Proof: We follow the proof from [9] of the case $\lambda = 1$, replacing an appeal to (1) in that proof with an appeal to Lemma 2.2.

Fix an integer $0 < \varphi < d$. For an independent set $I \in \mathcal{I}(G)$, recursively construct sets $T(I)$ and $D(T)$ as follows. Pick $u_1 \in I$ and set $T_1 = \{u_1\}$. Given $T_m = \{u_1, \ldots, u_m\}$, if there is $u_{m+1} \in I$ with $N(u_{m+1}) \setminus N(T_m) \geq \varphi$, then set $T_{m+1} = \{u_1, \ldots, u_{m+1}\}$. If there is no such u_{m+1}, then set $T = T_m$ and

$$D(T) = \{v \in V(G) \setminus N(T) : N(v) \setminus N(T) < \varphi\}.$$

Note that

$$|T| \leq \frac{N}{\varphi}, \tag{8}$$

since each step in the construction of T removes at least φ vertices from consideration; that

$$I \subseteq D \tag{9}$$

since if $I \setminus D \neq \emptyset$, the construction of T would not have stopped (note that $N(T) \cap I = \emptyset$); and that

$$|D| \leq \frac{Nd}{2d - \varphi}. \tag{10}$$

To see (10), consider the bipartite graph with partition classes D and $N(T)$ and edges induced from G. This graph has at most $d|N(T)| \leq d(N - |D|)$ edges (since each vertex in $N(T)$ has at most d edges to D, and there are at most $N - |D|$ such vertices), and at least $(d - \varphi)|D|$ edges (since each vertex in D has at least $d - \varphi$ edges to $N(T)$). Putting these two inequalities together gives (10).

Combining (8), (9) and (10) we see that we can construct all $I \in \mathcal{I}(G)$ by first picking a $T \subseteq V$ of size at most N/φ, next constructing $D(T)$, and finally generating all independent sets of the subgraph of G induced by $D(T)$. This graph inherits from G the property that all independent sets have size at most α, so using Lemma 2.2 it follows that

$$P(\lambda, G) \leq \sum_{t \leq N/\varphi} \binom{N}{t} \left(1 + \frac{\lambda N d}{(2d - \varphi)\alpha}\right)^{\alpha}. \tag{11}$$

We bound

$$1+\frac{\lambda N d}{(2d-\varphi)\alpha}=1+\frac{\lambda N}{2\alpha}\left(\frac{2d}{2d-\varphi}\right)\leq\left(1+\frac{\lambda N}{2\alpha}\right)\left(\frac{2d}{2d-\varphi}\right)$$

so that

$$\left(1+\frac{\lambda N d}{(2d-\varphi)\alpha}\right)^{\alpha}\leq\left(1+\frac{\lambda N}{2\alpha}\right)^{\alpha}\left(\frac{2d}{2d-\varphi}\right)^{N} \tag{12}$$

and naively bound

$$\sum_{t\leq N/\varphi}\binom{N}{t}\leq\left(\frac{N}{\varphi}+1\right)\binom{N}{N/\varphi}\leq\exp_2\left\{\frac{3N\log(e\varphi)}{\varphi}\right\} \tag{13}$$

(the second inequality mainly using $\binom{n}{k}\leq(en/k)^k$). Taking $\varphi=[\sqrt{d\log d}]$ the bounds in (11), (12) and (13) combine to yield

$$P(\lambda,G)\leq\left(1+\frac{\lambda N}{2\alpha}\right)^{\alpha}\exp_2\left\{cN\sqrt{\frac{\log d}{d}}\right\}$$

for some $c>0$, as claimed. $\square$

3 Proof of Theorem 1.3

We assume throughout that $d\geq 2$, since the theorem is straightforward for $d=1$. We begin by considering those G for which

$$\alpha(G)\geq\frac{N}{2}\left(1-C_\lambda\sqrt{\frac{\log d}{d}}\right),$$

where C_λ is a constant that will be determined later. In this case we use Lemma 2.1. For each v with $p_\prec(v)=0$ we have $2(1+\lambda)^{p_\prec(v)}-1=1$ and so if $\prec$ begins by listing the vertices of an independent set I then

$$\begin{aligned}P(\lambda,G) &\leq \prod_{v\in V(G)\setminus I}\left(2(1+\lambda)^{p_\prec(v)}-1\right)^{\frac{1}{d}}\\ &\leq (1+\lambda)^{\frac{1}{d}\sum_{v\in V(G)\setminus I}p_\prec(v)}\exp_2\left\{\frac{|V(G)\setminus I|}{d}\right\}\\ &= (1+\lambda)^{\frac{N}{2}}\exp_2\left\{\frac{N-|I|}{d}\right\}.\end{aligned} \tag{14}$$

Choosing I to be an independent set of size $\alpha(G)$ we get from (14) that

$$P(\lambda,G)\leq(1+\lambda)^{\frac{N}{2}}\exp_2\left\{\frac{N}{2d}\left(1+C_\lambda\sqrt{\frac{\log d}{d}}\right)\right\}. \tag{15}$$

We use Lemma 2.3 to bound $P(\lambda, G)$ in the case when $\alpha(G)$ satisfies

$$\alpha(G) < \frac{N}{2}\left(1 - C_\lambda\sqrt{\frac{\log d}{d}}\right).$$

For typographical convenience, in what follows we write x for $C_\lambda\sqrt{\log d/d}$. Since $\left(1+\frac{\lambda N}{2\alpha}\right)^\alpha$ is increasing in $\alpha > 0$ for $\lambda, N > 0$ we have

$$\begin{aligned}\left(1+\frac{\lambda N}{2\alpha}\right)^\alpha &\leq \left(1+\frac{\lambda}{1-x}\right)^{\frac{N(1-x)}{2}}\\ &= (1+\lambda)^{\frac{N}{2}}\left(1+\frac{x\lambda}{(1-x)(1+\lambda)}\right)^{\frac{N(1-x)}{2}}(1+\lambda)^{-\frac{Nx}{2}}\\ &\leq (1+\lambda)^{\frac{N}{2}}\exp\left\{\frac{Nx}{2}\left(\frac{\lambda}{1+\lambda}-\ln(1+\lambda)\right)\right\}.\end{aligned}$$

Since $\ln(1+\lambda) > \frac{\lambda}{1+\lambda}$ for all $\lambda > 0$ the exponent above is negative. By choosing $C_\lambda > 0$ to satisfy

$$\left(\ln(1+\lambda) - \frac{\lambda}{1+\lambda}\right)\frac{C_\lambda}{2\ln 2} = c$$

where c is the constant appearing in the bound in Lemma 2.3, we have

$$P(\lambda, G) \leq (1+\lambda)^{\frac{N}{2}} \tag{16}$$

in this case. Combining (15) and (16) we obtain

$$P(\lambda, G) \leq (1+\lambda)^{\frac{N}{2}}\exp_2\left\{\frac{N}{2d}\left(1+C_\lambda\sqrt{\frac{\log d}{d}}\right)\right\}$$

for all G, completing the proof of Theorem 1.3.

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