Total non-negativity of some combinatorial matrices

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Abstract

Many combinatorial matrices — such as those of binomial coefficients, Stirling numbers of both kinds, and Lah numbers — are known to be totally non-negative, meaning that all minors (determinants of square submatrices) are non-negative.

The examples noted above can be placed in a common framework: for each one there is a non-decreasing sequence $(a_1, a_2, \ldots)$, and a sequence $(e_1, e_2, \ldots)$, such that the (m, k)-entry of the matrix is the coefficient of the polynomial $(x - a_1) \cdots (x - a_k)$ in the expansion of $(x - e_1) \cdots (x - e_m)$ as a linear combination of the polynomials $1, x - a_1, \ldots, (x - a_1) \cdots (x - a_m)$.

We consider this general framework. For a non-decreasing sequence $(a_1, a_2, \ldots)$ we establish necessary and sufficient conditions on the sequence $(e_1, e_2, \ldots)$ for the corresponding matrix to be totally non-negative. As an application we obtain total non-negativity of a family of matrices associated with chordal graphs.

1 Introduction

A matrix — finite or infinite — is *totally non-negative* if all minors (determinants of square sub-matrices) are non-negative. Totally non-negative matrices occur frequently in combinatorics and have been the subject of much investigation. See e.g. [2, 11, 12, 18] for an overview. Here are a few of the most prominent examples:

- $\left[\binom{m}{k}\right]_{m,k\geq 0}$, where $\binom{m}{k}$ is the usual binomial coefficient;
- $\left[\left\{ {m \atop k} \right\}\right]_{m,k\geq 0}$, where $\left\{ {m \atop k} \right\}$ is the *Stirling number of the second kind*, counting partitions of a set of size m into k non-empty blocks;
- $\left[\left[{m \atop k} \right]\right]_{m,k\geq 0}$, where $\left[{m \atop k} \right]$ is the *(unsigned) Stirling number of the first kind*, counting partitions of a set of size m into k non-empty cyclically ordered blocks; and
- $[L(m, k)]_{m,k\geq 0}$, where $L(m, k)$ is the *Lah number*, counting partitions of a set of size m into k non-empty linearly ordered blocks.

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These examples can be placed in a common framework. Given two real sequences $\mathbf{a} = (a_1, a_2, \ldots)$ and $\mathbf{e} = (e_1, e_2, \ldots)$, either both infinite or both finite and of the same length, define a matrix $S^{\mathbf{a},\mathbf{e}} = (S^{\mathbf{a},\mathbf{e}}(m,k))_{m,k\geq 0}$ via the relations

$$\prod_{i=1}^{m}(x - e_i) = \sum_{k=0}^{m} S^{\mathbf{a},\mathbf{e}}(m,k) \prod_{i=1}^{k}(x - a_i) \tag{1}$$

for $m \geq 0$. If $\mathbf{a}$ and $\mathbf{e}$ are infinite then $S^{\mathbf{a},\mathbf{e}}$ is infinite with rows and columns indexed by $\{0, 1, 2, \ldots\}$, while if $\mathbf{a}$ and $\mathbf{e}$ are both of length n then $S^{\mathbf{a},\mathbf{e}}$ is $(n+1)$ by $(n+1)$ with rows and columns indexed by $\{0, 1, \ldots, n\}$. Note that (1) uniquely determines $S^{\mathbf{a},\mathbf{e}}(m,k)$ for each $m, k \geq 0$.

Let us see how the four examples given earlier fit into this general framework:

- Taking $e_i = -1$ and $a_i = 0$ for all i yields $S^{\mathbf{a},\mathbf{e}}(m,k) = \binom{m}{k}$;
- taking $e_i = 0$ and $a_i = i - 1$ for all i yields $S^{\mathbf{a},\mathbf{e}}(m,k) = \left\{ {m \atop k} \right\}$ via the identity

$$x^m = \sum_{k\geq 0} \left\{ {m \atop k} \right\} x(x-1)\cdots(x-(k-1)) \tag{2}$$

  for $m \geq 0$;
- taking $e_i = -(i-1)$ and $a_i = 0$ for all i yields $S^{\mathbf{a},\mathbf{e}}(m,k) = \left[{m \atop k} \right]$ via $x(x+1)\cdots(x+(m-1)) = \sum_{k\geq 0} \left[{m \atop k} \right] x^k$ for $m \geq 0$; and
- taking $e_i = -(i-1)$ and $a_i = i-1$ for all i yields $S^{\mathbf{a},\mathbf{e}}(m,k) = L(m,k)$ via $x(x+1)\cdots(x+(m-1)) = \sum_{k\geq 0} L(m,k)x(x-1)\cdots(x-(k-1))$ for $m \geq 0$.

The main result of this note is a characterization, for each non-decreasing sequence $\mathbf{a}$, of those sequences $\mathbf{e}$ such that the matrix $S^{\mathbf{a},\mathbf{e}}$ is totally non-negative.

Definition 1.1. If $\mathbf{a}$ is non-decreasing, we say that $\mathbf{e}$ is a *restricted growth sequence* relative to $\mathbf{a}$ if for each $i \geq 1$ it holds that $e_i \leq a_{f(i)}$, where $f(1) = 1$ and for $i \geq 1$

$$f(i+1) = \begin{cases} f(i) & \text{if } e_i < a_{f(i)} \\ f(i)+1 & \text{if } e_i = a_{f(i)}. \end{cases}$$

Informally, each e_i is at most a certain cap. The cap for e_1 is a_1, and it continues to be a_1 until the first time that an entry of $\mathbf{e}$ takes the value a_1, at which point it becomes a_2, and so on. If $\mathbf{a} = (0, 1, \ldots, n-1, \ldots)$ then a non-negative integral sequence $\mathbf{e}$ is a restricted growth sequence relative to $\mathbf{a}$ exactly if it is a restricted growth sequence in the usual sense, that is, one satisfying $e_1 = 0$ and $e_{i+1} \leq 1 + \max_{j=1,\ldots,i} e_j$ for $i \geq 1$.

Notice that in all the examples above, $\mathbf{a}$ was non-decreasing and $\mathbf{e}$ was a restricted growth sequence relative to $\mathbf{a}$. The total non-negativity of the matrices arising from these examples thus follows from the following result.

Theorem 1.2. *Let* $\mathbf{a}$ *be a non-decreasing sequence. Then*

1. *the matrix $S^{\mathbf{a},\mathbf{e}}$ is totally non-negative if and only if* $\mathbf{e}$ *is a restricted growth sequence relative to* $\mathbf{a}$*, and*

2. *if* $\mathbf{e}$ *is* not *a restricted growth sequence relative to* $\mathbf{a}$ *then the failure of $S^{\mathbf{a},\mathbf{e}}$ to be totally non-negative is witnessed by a negative entry in $S^{\mathbf{a},\mathbf{e}}$.*

The proof of Theorem 1.2 involves producing a weighted planar network whose path-matrix is $S^{\mathbf{a},\mathbf{e}}$, and then appealing to Lindström's lemma (see Section 3 for details). The network that we initially produce, however, does not have all non-negative entries, precluding an immediate application of Lindström. A substantial part of the proof involves carefully modifying the weights of the initial network to remove the negative entries, without changing the associated path matrix.

We prove Theorem 1.2 in Section 3. Before that, in Section 2, we consider an application to graph Stirling numbers of chordal graphs.

To conclude the introduction, let us observe that the numbers $S^{\mathbf{a},\mathbf{e}}(m,k)$ defined in (1) satisfy the recurrence

$$S^{\mathbf{a},\mathbf{e}}(m,k) = S^{\mathbf{a},\mathbf{e}}(m-1,k-1) + (a_{k+1} - e_m)S^{\mathbf{a},\mathbf{e}}(m-1,k) \quad \text{for } m,k > 0 \tag{3}$$

with initial conditions $S^{\mathbf{a},\mathbf{e}}(0,0) = 1$, $S^{\mathbf{a},\mathbf{e}}(0,k) = 0$ for $k > 0$ and $S^{\mathbf{a},\mathbf{e}}(m,0) = \prod_{i=1}^{m}(a_1 - e_i)$ for $m > 0$ (we prove this in Section 3, see (7)). A number of authors have considered the question of total non-negativity of matrices $(a_{m,k})_{m,k\geq 0}$ with the $a_{m,k}$ defined via recurrences similar to (3). Brenti [2], for example, considered the recurrence $a_{m,k} = z_m a_{m-t,k-1} + y_m a_{m-1,k-1} + x_m a_{m-1,k}$ ($t \in \mathbb{N}$). More recently Chen, Liang and Wang [4, 5] considered $a_{m,k} = r_k a_{m-1,k-1} + s_k a_{m-1,k} + t_{k+1} a_{m-1,k+1}$ and also the more general situation where the $a_{m,k}$'s form a Riordan array. The recurrence (3) does not seem to fit these settings.

2 Graph Stirling numbers of chordal graphs

The Stirling numbers of the second kind have a natural generalization to the setting of graphs. For a graph G and an integer k, the *graph Stirling number of the second kind* $\left\{ {G \atop k} \right\}$ is the number of ways of partitioning the vertex set of G into k non-empty independent sets (an *independent set* being a set of pairwise non-adjacent vertices). This is indeed a generalization, since if E_m is the graph on m vertices with no edges, then $\left\{ {E_m \atop k} \right\} = \left\{ {m \atop k} \right\}$.

This notion of graph Stirling number of the second kind was probably first introduced by Tomescu [19] and was subsequently reintroduced by numerous authors including Korfhage [14], Goldman, Joichi and White [13] and Duncan and Peele [7]. Its properties have been well studied, see for example [1, 3, 6, 8, 10, 16, 17].

The Stirling number of the first kind does not have such a natural graph analog. In [9] Eu, Fu, Liang and Wong present a notion of a graph Stirling number of the first kind for the family of quasi-threshold graphs, based on generalizations of the relation $x^m D^m = \sum_{k\geq 0}(-1)^{m-k}\left[{m \atop k} \right](xD)^k$ in the Weyl algebra on symbols x and D (the algebra over the reals generated by the relation $Dx = xD + 1$).

Here we take a different approach. It is well-known that the inverse of the matrix of Stirling numbers of the second kind is the matrix of *signed* Stirling numbers of the first kind:

$$\left[\left\{ m \atop k \right\}\right]_{m,k\geq 0}^{-1} = \left[(-1)^{m-k}\left[m \atop k \right]\right]_{m,k\geq 0}.$$

This suggests the following. For a graph G on n vertices, ordered $v_1, \dots, v_n$, let G_m denote the subgraph of G induced by $v_1, \dots, v_m$, and consider the matrices

$$S_G = \left[\left\{ G_m \atop k \right\}\right]_{m,k=0}^{n} \quad \text{and} \quad s_G = S_G^{-1}.$$

When $G = E_n$ the (m,k)-entry of s_G is $(-1)^{m-k}\left[m \atop k \right]$. It is easy to find examples of graphs such that however the vertices are ordered the matrix s_G does not have the checkerboard sign pattern exhibited by the inverse of $[\left\{ m \atop k \right\}]_{m,k\geq 0}$, meaning that this approach may not be suited to defining a notion of graph Stirling number of the first kind for all graphs.

However, there is a class of graphs which admit a natural ordering of the vertices with respect to which the matrix s_G has a checkerboard sign-pattern, that is, with the (m,k)-entry having sign $(-1)^{m-k}$. A *chordal graph* is a graph in which every cycle of length four or greater has a chord, i.e., it is a graph that contains no induced cycles of length four or greater. A useful characterization of chordal graphs is that G is chordal if and only if it is possible to order the vertices as $v_1, \dots, v_n$ so that for each $m \in \{1, \dots, n\}$ the neighbors of v_m among $v_1, \dots, v_{m-1}$ induce a clique (see e.g. [20, Section 5.3]). Such an ordering is referred to as a *perfect elimination order*.

Theorem 2.1. *Let G be a chordal graph with perfect elimination order $v_1, \dots, v_n$, and let G_m be the subgraph of G induced by $v_1, \dots, v_m$. Let $S_G = \left[\left\{ G_m \atop k \right\}\right]_{m,k=0}^{n}$ and $s_G = S_G^{-1}$. For all m, k the (m,k)-entry of s_G has sign $(-1)^{m-k}$.*

A stronger result than Theorem 2.1 holds. Notice that the matrix S_G has determinant 1 and so by Cramer's rule the (m,k)-entry of the inverse is $(-1)^{m-k}$ times the determinant of the $n-1$ by $n-1$ minor obtained from S_G by deleting the kth row and the mth column. It follows that if S_G is totally non-negative then the (m,k)-entry of s_G has sign $(-1)^{m-k}$, and so the following result generalizes Theorem 2.1.

Theorem 2.2. *Let G be a chordal graph with perfect elimination order $v_1, \dots, v_n$, and let G_m be the subgraph of G induced by $v_1, \dots, v_m$. Let $S_G = \left[\left\{ G_m \atop k \right\}\right]_{m,k=0}^{n}$. Then S_G is totally non-negative.*

As we will now see, Theorem 2.2 is a special case of Theorem 1.2. The *chromatic polynomial* $\chi_G(x)$ of a graph G is the polynomial in x whose value at positive integers x is the number of ways of coloring G from a palette of x colors in such a way that adjacent vertices receive distinct colors. That $\chi_G(x)$ is indeed a polynomial in x follows from the following identity: for G a graph on m vertices,

$$\chi_G(x) = \sum_{k=0}^{m} \left\{ G \atop k \right\} x(x-1)\cdots(x-(k-1)). \tag{4}$$

Indeed, one way to enumerate the colorings of G from a palette of x colors in such a way that adjacent vertices receive distinct colors is to first specify k, the number of colors used, then specify a partition of the vertex set of G into k non-empty independent sets ($\left\{ {G \atop k} \right\}$ options), which will be the color classes, and finally specify the colors that appear on each of the classes ($x(x-1)\cdots(x-(k-1))$ options). (Observe that by taking G to be the graph on m vertices with no edges we recover (2) from (4)).

For a chordal graph G with perfect elimination order $v_1, \ldots, v_n$, for $i \geq 1$ denote by $e_i = e_i(G)$ the number of neighbors that v_i has among $v_1, \ldots, v_{i-1}$. We have that $\chi_{G_m}(x) = (x-e_1)(x-e_2)\cdots(x-e_m)$ (coloring the vertices of G_m sequentially from v_1 to v_m, at the step when v_j is colored all colors are available except those used on the e_j neighbors of v_j among $\{v_1, \ldots, v_{j-1}\}$; since these neighbors form a clique, between them they account for e_j colors, leaving $x - e_j$ available for v_j). Thus, in light of (4), if we knew that $(e_1, \ldots, e_n)$ formed a restricted growth sequence relative to $(0, 1, \ldots, n-1)$, then the total non-negativity of $\left(\left\{ {G_m \atop k} \right\}\right)_{m,k=0}^{n}$ would follow from Theorem 1.2.

In fact, we have the following.

Claim 2.3. *Let G be a chordal graph G with perfect elimination order $v_1, \ldots, v_n$. Defining $e_i = e_i(G)$ as above, we have that $(e_1, \ldots, e_n)$ is a restricted growth sequence relative to $(0, 1, \ldots, n-1)$. Moreover, if $(e'_1, \ldots, e'_n)$ is any restricted growth sequence relative to $(0, 1, \ldots, n-1)$ then there is a chordal graph G with perfect elimination order $v_1, \ldots, v_n$ such that $e_i(G) = e'_i$ for all $i \leq n$.*

Proof: We begin by showing that $(e_1, \ldots, e_n)$ is a restricted growth sequence relative to $(0, 1, \ldots, n-1)$. Certainly $e_1 = 0$. Now consider vertex v_k for $k > 1$. It is adjacent to e_k vertices among $v_1, \ldots, v_{k-1}$, with the largest of these (in the ordering $v_1 < v_2 < \cdots$) being, say, v_j. Because v_k forms a clique with its neighbors among $v_1, \ldots, v_{k-1}$, it follows that v_j has at least $e_k - 1$ neighbors among $v_1, \ldots, v_{j-1}$, so $e_j \geq e_k - 1$. From this it follows that $e_k \leq e_j + 1 \leq 1 + \max_{i<k} e_i$, exactly the condition that says that $(e_1, \ldots, e_n)$ is a restricted growth sequence relative to $(0, 1, \ldots, n-1)$.

Next suppose $(e'_1, \ldots, e'_n)$ is a restricted growth sequence relative to $(0, 1, \ldots, n-1)$. We construct a chordal graph G with perfect elimination order $v_1, \ldots, v_n$ such that $e_i(G) = e'_i$ for all $i = 1, \ldots, n$, inductively, starting with an isolated vertex v_1. Suppose that the adjacency structure among $v_1, \ldots, v_{k-1}$ has been determined. We have that $e_k \leq 1 + \max_{i<k} e_i$, which means that (by induction) among $v_1, \ldots, v_{k-1}$ there are some e_k vertices that form a clique. The construction can be continued by joining v_k to any such e_k vertices. □

3 Proof of Theorem 1.2

A key tool will be the following explicit expression for $S^{\mathbf{a},\mathbf{e}}(m, k)$.

Lemma 3.1. *For arbitrary* **a** *and* **e** *we have*

$$S^{\mathbf{a},\mathbf{e}}(m, k) = \sum_{\substack{S=\{s_1,\ldots,s_{m-k}\}\subseteq\{1,\ldots,m\} \\ s_1<\ldots<s_{m-k}}} \prod_{i=1}^{m-k} (a_{s_i-i+1} - e_{s_i}), \tag{5}$$

and, equivalently, denoting the (m,k)*-entry of* $(S^{\mathbf{a},\mathbf{e}})^{-1}$ *by* $s^{\mathbf{a},\mathbf{e}}(m,k)$*,*

$$(-1)^{m-k} s^{\mathbf{a},\mathbf{e}}(m,k) = \sum_{\substack{S=\{s_1,\ldots,s_{m-k}\}\subseteq\{1,\ldots,m\} \\ s_1<\ldots<s_{m-k}}} \prod_{i=1}^{m-k} (a_{s_i} - e_{s_i-i+1}). \tag{6}$$

Notice that in the chordal graph setting e_{s_i-i+1} is the number of edges from vertex v_{s_i-i+1} to earlier vertices, so is at most s_i-i, which is at most s_i-1, which is a_{s_i}, and so the quantity on the right-hand side of the formula for $(-1)^{m-k}s(m,k)$ is non-negative. This establishes directly that the sign of the (m,k)-entry of s_G is $(-1)^{m-k}$, as asserted by Theorem 2.1.

In the sequel we will only give a proof of (5), so here let us note that this identity implies (6). Indeed, from (1) we have that $S^{\mathbf{a},\mathbf{e}}(m,k)$ is the coefficient of $(x-a_1)\cdots(x-a_k)$ in the unique expansion of $(x-e_1)\cdots(x-e_m)$ as a linear combination of $1, x-a_1, \ldots, (x-a_1)\cdots(x-a_m)$, and so from basic linear algebra considerations we see that the $s^{\mathbf{a},\mathbf{e}}(m,k)$ are uniquely determined by the relations

$$\prod_{i=1}^{m}(x-a_i) = \sum_{k=0}^{m} s^{\mathbf{a},\mathbf{e}}(m,k) \prod_{i=1}^{k}(x-e_i)$$

for $m \geq 0$. A direct application of (5) yields

$$\begin{aligned}(-1)^{m-k} s^{\mathbf{a},\mathbf{e}}(m,k) &= (-1)^{m-k} \sum_{\substack{S=\{s_1,\ldots,s_{m-k}\}\subseteq\{1,\ldots,m\} \\ s_1<\ldots<s_{m-k}}} \prod_{i=1}^{m-k} (e_{s_i-i+1} - a_{s_i}) \\ &= \sum_{\substack{S=\{s_1,\ldots,s_{m-k}\}\subseteq\{1,\ldots,m\} \\ s_1<\ldots<s_{m-k}}} \prod_{i=1}^{m-k} (a_{s_i} - e_{s_i-i+1}).\end{aligned}$$

Of course the same argument in reverse shows that the two identities are equivalent.

Proof (of Lemma 3.1): We will show that both sides of (5) satisfy the same recurrence relation and initial conditions. To that end write $f(m,k)$ for the expression on the right-hand side of (5). We begin by establishing some boundary values for $f(m,k)$.

- We have $f(0,0) = 1$ (the sum has one summand, associated with $S = \emptyset$, and this summand is the empty product and so has value 1), and more generally $f(m,m) = 1$ for all m.
- For $m > 0$, $f(m,0) = (a_1 - e_1)\cdots(a_1 - e_m)$.
- For $k > 0$, $f(0,k) = 0$ (the sum defining f in this case is empty), and more generally for $k > m$, $f(m,k) = 0$.

Next we establish a recurrence for $f(m,k)$. For $m > k > 0$ we have

$$f(m,k) = f(m-1,k-1) + (a_{k+1} - e_m) f(m-1,k).$$

The terms on the right-hand side here come from considering first those S with $m \notin S$ and then those with $m \in S$; in the latter case m is always the greatest element of S and so contributes a factor $a_{m-(m-k)+1} - e_m = a_{k+1} - e_m$ to each summand.

Next consider the quantity $S^{\mathbf{a},\mathbf{e}}(m,k)$. We easily have $S^{\mathbf{a},\mathbf{e}}(0,0) = 1$, and more generally $S^{\mathbf{a},\mathbf{e}}(m,m) = 1$ for all m, as well as $S^{\mathbf{a},\mathbf{e}}(m,0) = (a_1 - e_1)\cdots(a_1 - e_m)$ for $m > 0$ (evaluate both sides of (1) at $x = a_1$). We also have $S^{\mathbf{a},\mathbf{e}}(0,k) = 0$ for $k > 0$ and more generally $S^{\mathbf{a},\mathbf{e}}(m,k) = 0$ for $k > m$. We also have the recurrence

$$S^{\mathbf{a},\mathbf{e}}(m,k) = S^{\mathbf{a},\mathbf{e}}(m-1,k-1) + (a_{k+1} - e_m)S^{\mathbf{a},\mathbf{e}}(m-1,k) \tag{7}$$

for $m > k > 0$. To verify this, consider the expression

$$\begin{gathered} S^{\mathbf{a},\mathbf{e}}(m,0) + S^{\mathbf{a},\mathbf{e}}(m-1,m-1)(x-a_1)\cdots(x-a_m) \\ +\sum_{k=1}^{m-1}\left(S^{\mathbf{a},\mathbf{e}}(m-1,k-1) + (a_{k+1}-e_m)S^{\mathbf{a},\mathbf{e}}(m-1,k)\right)(x-a_1)\cdots(x-a_k) \end{gathered} \tag{8}$$

(a linear combination of the polynomials $1, x-a_1, \ldots, (x-a_1)\cdots(x-a_m)$). Rearranging terms (8) becomes

$$\begin{gathered} S^{\mathbf{a},\mathbf{e}}(m,0) + S^{\mathbf{a},\mathbf{e}}(m-1,0)(x-a_1) \\ +(x-e_m)\sum_{k=1}^{m-1} S^{\mathbf{a},\mathbf{e}}(m-1,k)(x-a_1)\cdots(x-a_k). \end{gathered} \tag{9}$$

Writing $x - a_1 = (x-e_m) - (a_1 - e_m)$ in the second term of (9) yields

$$\begin{gathered} S^{\mathbf{a},\mathbf{e}}(m,0) - S^{\mathbf{a},\mathbf{e}}(m-1,0)(a_1-e_m) \\ +(x-e_m)\sum_{k=0}^{m-1} S^{\mathbf{a},\mathbf{e}}(m-1,k)(x-a_1)\cdots(x-a_k). \end{gathered} \tag{10}$$

Via the initial conditions the first two terms of (10) sum to 0, and via the defining relation for $S^{\mathbf{a},\mathbf{e}}(m-1,\cdot)$ ((1) with m replaced by $m-1$) the remaining terms sum to $\prod_{i=1}^{m}(x-e_i)$. The recurrence (7) now follows from (1) via linear algebra considerations.

Since $f(m,k)$ and $S^{\mathbf{a},\mathbf{e}}(m,k)$ satisfy the same recurrence and initial conditions, they are equal. □

Lemma 3.1 allows us to express $S^{\mathbf{a},\mathbf{e}}(m,k)$ in terms of complete symmetric and elementary symmetric functions. Recall that $h_\ell(x_1,\ldots,x_t)$ is the degree ℓ complete symmetric polynomial in $x_1,\ldots,x_t$ (the sum of all degree ℓ monomials with coefficients 1) and $s_\ell(x_1,\ldots,x_t)$ is the degree ℓ elementary symmetric polynomial in $x_1,\ldots,x_t$ (the sum of all degree ℓ linear monomials with coefficients 1); so, for example, $h_2(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$ while $s_2(x_1,x_2,x_3) = x_1x_2 + x_1x_3 + x_2x_3$.

Lemma 3.2. *For arbitrary* $\mathbf{a}$ *and* $\mathbf{e}$*,*

$$S^{\mathbf{a},\mathbf{e}}(m,k) = \sum_{\ell=0}^{m-k}(-1)^\ell h_{m-k-\ell}(a_1,\ldots,a_{k+1})s_\ell(e_1,\ldots,e_m). \tag{11}$$

Proof: One possible approach is to show that that the expressions on the right-hand sides of (5) and (11) are equal. This can be achieved by noting that when the right-hand side of (5) is expanded as a polynomial in the e_i's, the monomials that arise are precisely the linear monomials in $e_1,\ldots,e_m$. For a given ℓ, $0\le\ell\le m-k$, and $T=\{t_1,\ldots,t_\ell\}\subseteq\{1,\ldots,m\}$ with $t_1<\ldots<t_\ell$, the coefficient of $e_{t_1}\ldots e_{t_\ell}$ turns out to be $(-1)^\ell h_{m-k-\ell}(a_1,\ldots,a_{k+1})$ (independent of the particular choice of T); this proves the lemma.

We take instead a linear algebra approach. From (1) we have

$$\prod_{i=1}^{m}(x-e_i)=\sum_k S^{\mathbf{a},\mathbf{e}}(m,k)\prod_{i=1}^{k}(x-a_i)$$

(where the sum runs over all integers k, although the summand will only be non-zero for $k\in\{0,1,\ldots,m\}$). It follows that

$$\begin{aligned}\prod_{i=1}^{m}(x-e_i) &= \sum_j S^{\mathbf{0},\mathbf{e}}(m,j)x^j\\ &= \sum_j S^{\mathbf{0},\mathbf{e}}(m,j)\sum_k S^{\mathbf{a},\mathbf{0}}(j,k)\prod_{i=1}^{k}(x-a_i)\\ &= \sum_k\left(\sum_j S^{\mathbf{0},\mathbf{e}}(m,j)S^{\mathbf{a},\mathbf{0}}(j,k)\right)\prod_{i=1}^{k}(x-a_i)\end{aligned}$$

so that

$$S^{\mathbf{a},\mathbf{e}}(m,k)=\sum_j S^{\mathbf{0},\mathbf{e}}(m,j)S^{\mathbf{a},\mathbf{0}}(j,k).$$

Now Lemma 3.1 gives

$$\begin{aligned}S^{\mathbf{0},\mathbf{e}}(m,j) &= \sum_{\substack{S=\{s_1,\ldots,s_{m-j}\}\subseteq\{1,\ldots,m\}\\ s_1<\ldots<s_{m-j}}}\prod_{i=1}^{m-j}(-e_{s_i})\\ &= (-1)^{m-j}s_{m-j}(e_1,\ldots,e_m)\end{aligned}$$

and

$$\begin{aligned}S^{\mathbf{a},\mathbf{0}}(j,k) &= \sum_{\substack{S=\{s_1,\ldots,s_{j-k}\}\subseteq\{1,\ldots,j\}\\ s_1<\ldots<s_{j-k}}}\prod_{i=1}^{j-k}a_{s_i-i+1}\\ &= h_{j-k}(a_1,\ldots,a_{k+1}).\end{aligned}$$

Combining these two equations, and re-indexing via $\ell=m-j$, leads to (11).

□

We now require some well-known results from the theory of totally non-negative matrices. Consider the weighted directed planar network shown in Figure 1, where horizontal lines are oriented to the right and vertical lines upward (so that $s_0,\ldots,s_n$ are sources and $t_0,\ldots,t_n$ are sinks), and all horizontal edge weights are 1.

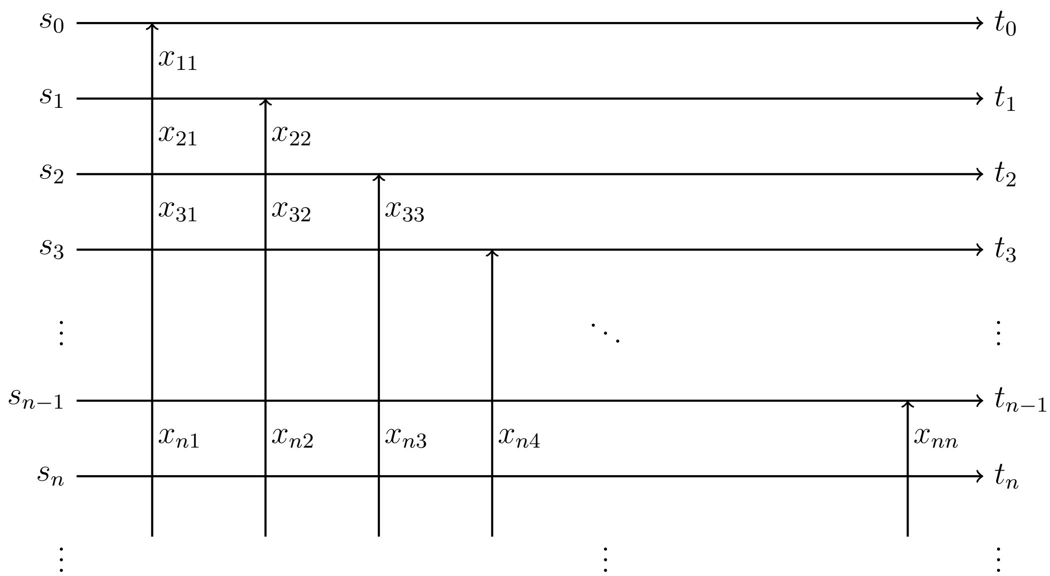

Figure 1: A weighted directed planar network

In the sequel we will represent this network by its (doubly infinite) array of weights, viz:

$$\begin{array}{lllllll}
x_{11} & & & & & & \\
x_{21} & x_{22} & & & & & \\
x_{31} & x_{32} & x_{33} & & & & \\
x_{41} & x_{42} & x_{43} & x_{44} & & & \\
x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & & \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \\
x_{n1} & x_{n2} & x_{n3} & x_{n4} & \cdots & x_{n(n-1)} & x_{nn} \\
\vdots & \vdots & \vdots & \vdots & & & & \ddots,
\end{array}$$

and we will refer to the location of the weight x_{mk} in this array as the $[m, k]$-*position* of the array (using square brackets to distinguish this from an entry in a matrix).

By the *path-matrix* of the planar network we mean the doubly infinite matrix whose (i, j)-entry (with rows and columns indexed by $\{0, 1, 2, \ldots\}$) is the sum of the weights of all the directed paths from s_i to t_j, where the weight of one such path is the product, over all edges traversed, of the weight of the edge. For example, the $(3, 1)$-entry of the path-matrix of the planar network in Figure 1 is $x_{31}x_{21} + x_{31}x_{22} + x_{32}x_{22}$. Notice that the path-matrix of the network in Figure 1 is lower-triangular with 1's down the main diagonal. The following [15] (see also, for example, [18]), is a standard result from the theory of totally non-negative matrices.

Lemma 3.3. *(Lindström's Lemma) If the matrix M is the path-matrix of a planar network with all non-negative weights, then M is totally non-negative.*

Indeed, the minor corresponding to selecting the rows indexed by I and columns indexed by J (with indexing of rows and columns starting from 0) equals the sum of the weights of all the collections of $|I|$ vertex disjoint paths from the sources $\{s_i : i \in I\}$ to the sinks $\{t_j : j \in J\}$, where the weight of a collection of paths is the product of the weights of the individual paths in the collection; but we will not need this.

So, to prove item 1 of Theorem 1.2, it suffices to produce, for each $\mathbf{a}$ and $\mathbf{e}$ with $\mathbf{a}$ non-decreasing and $\mathbf{e}$ a restricted growth sequence relative to $\mathbf{a}$, a planar network all of whose weights are non-negative and whose path-matrix is $S^{\mathbf{a},\mathbf{e}}$. We will achieve the construction of this network in stages, first producing a network whose path-matrix is $S^{\mathbf{a},\mathbf{e}}$ but which may have some negative entries, and then modifying it in a way that makes all the negative entries non-negative, without changing the associated path-matrix.

Lemma 3.4. *For arbitrary* $\mathbf{a}$ *and* $\mathbf{e}$*, the following planar network has* $S^{\mathbf{a},\mathbf{e}}$ *as its path-matrix:*

$$\begin{array}{ccccccc} a_1-e_1 &&&&&& \\ a_1-e_2 & a_2-e_1 &&&&& \\ a_1-e_3 & a_2-e_2 & a_3-e_1 &&&& \\ a_1-e_4 & a_2-e_3 & a_3-e_2 & a_4-e_1 &&& \\ a_1-e_5 & a_2-e_4 & a_3-e_3 & a_4-e_2 & a_5-e_1 && \\ \vdots & \vdots & \vdots & \vdots && \ddots & \\ a_1-e_n & a_2-e_{n-1} & a_3-e_{n-2} & a_4-e_{n-3} & \cdots & a_{n-1}-e_2 & a_n-e_1 \\ \vdots & \vdots & \vdots & \vdots &&& \ddots \end{array} \tag{12}$$

Proof: Clearly the first row and column of the path-matrix, as well as the main diagonal and everything above the main diagonal, agree with $S^{\mathbf{a},\mathbf{e}}$, so we focus on $m > k \geq 1$. In this range the relevant portion of (12) (the only vertical edges that can be traversed in a path from s_m to t_k) is the following:

$$\begin{array}{ccccc} a_1-e_{k+1} & a_2-e_k & a_3-e_{k-1} & \cdots & a_{k+1}-e_1 \\ a_1-e_{k+2} & a_2-e_{k+1} & a_3-e_k & \cdots & a_{k+1}-e_2 \\ a_1-e_{k+3} & a_2-e_{k+2} & a_3-e_{k+1} & \cdots & a_{k+1}-e_3 \\ \vdots & \vdots & \vdots && \vdots \\ a_1-e_{m-1} & a_2-e_{m-2} & a_3-e_{m-3} & \cdots & a_{k+1}-e_{m-k-1} \\ a_1-e_m & a_2-e_{m-1} & a_3-e_{m-2} & \cdots & a_{k+1}-e_{m-k}. \end{array} \tag{13}$$

(Specifically these are the entries in the $[s,t]$-position for $k+1 \leq s \leq m$ and $1 \leq t \leq k+1$ — there are $m-k$ rows and $k+1$ columns).

Now consider the following network, which is obtained from (13) by mapping e_i to e_{m-i+1} for each i:

$$\begin{array}{ccccc} a_1-e_{m-k} & a_2-e_{m-k+1} & a_3-e_{m-k+2} & \cdots & a_{k+1}-e_m \\ a_1-e_{m-k-1} & a_2-e_{m-k} & a_3-e_{m-k+1} & \cdots & a_{k+1}-e_{m-1} \\ a_1-e_{m-k-2} & a_2-e_{m-k-1} & a_3-e_{m-k} & \cdots & a_{k+1}-e_{m-2} \\ \vdots & \vdots & \vdots && \vdots \\ a_1-e_2 & a_2-e_3 & a_3-e_4 & \cdots & a_{k+1}-e_{k+2} \\ a_1-e_1 & a_2-e_2 & a_3-e_3 & \cdots & a_{k+1}-e_{k+1}. \end{array} \tag{14}$$

A path from s_m to t_k in either network corresponds to a composition $b_1+\cdots+b_{k+1} = m-k$ of $m-k$ into $k+1$ non-negative parts, via: from s_m take b_1 vertical steps along the first column of the weight array, then one horizontal step, then b_2 vertical steps along the second column of the weight array, and so on.

In (14), the weight of the path corresponding to the composition $b_1+\cdots+b_{k+1}=m-k$ is a product of the form $\prod_{i=1}^{m-k}\left(a_{f(i)}-e_{g(i)}\right)$, where the sequence $(f(1),\ldots,f(m-k))$ consists of b_1 1's, followed by b_2 2's, and so on, and the sequence $(g(1),\ldots,g(m-k))$ starts $1,2,\ldots,b_1$, then moves on to an increasing sequence of consecutive integers of length b_2 starting from b_1+2, and so on; in other words, it is

$$\prod_{i=1}^{m-k}(a_{s_i-i+1}-e_{s_i})$$

where $\{s_1,\ldots,s_{m-k}\}=\{1,\ldots,b_1,\widehat{b_1+1},b_1+2,\ldots,b_1+b_2+1,\widehat{b_1+b_2+2},\ldots\}$ (the hats indicating missing elements). As $(b_1,\ldots,b_{k+1})$ runs over all $\binom{m}{m-k}$ compositions of $m-k$ into $k+1$ parts, the sets $\{s_1,\ldots,s_{m-k}\}$ run over all $\binom{m}{m-k}$ subsets of $\{1,\ldots,m\}$ of size $m-k$, and so we get that in (14) the sum of the weights of the paths from s_m to t_k is

$$\sum_{\substack{S=\{s_1,\ldots,s_{m-k}\}\subseteq\{1,\ldots,m\}\\ s_1<\ldots<s_{m-k}}}\prod_{i=1}^{m-k}(a_{s_i-i+1}-e_{s_i}). \tag{15}$$

By Lemma 3.1 the expression in (15) is $S^{\mathbf{a},\mathbf{e}}(m,k)$. The weights in (13) are obtained from those in (14) by a permutation of the e_i's, and by Lemma 3.2 the expression in (15) is invariant under such permutations. It follows that in (13) the sum of the weights of the paths from s_m to t_k is also $S^{\mathbf{a},\mathbf{e}}(m,k)$. □

Even if $\mathbf{a}$ is non-decreasing and $\mathbf{e}$ is a restricted growth sequence relative to $\mathbf{a}$, it may be that some of the entries of (12) are negative (a_1-e_2, for example). We now describe a transformation that iteratively turns the network (12) into one that has only non-negative weights, without changing the associated path matrix.

Lemma 3.5. *For arbitrary* $\mathbf{a}$ *and* $\mathbf{e}$*, if* $a_1=e_1$ *then the following planar network has* $S^{\mathbf{a},\mathbf{e}}$ *as its path-matrix:*

$$\begin{array}{cccccccc}
a_1-e_1 \\
a_1-e_1 & a_2-e_2 \\
a_1-e_1 & a_2-e_3 & a_3-e_2 \\
a_1-e_1 & a_2-e_4 & a_3-e_3 & a_4-e_2 \\
a_1-e_1 & a_2-e_5 & a_3-e_4 & a_4-e_3 & a_5-e_2 \\
\vdots & \vdots & \vdots & \vdots & & \ddots \\
a_1-e_1 & a_2-e_n & a_3-e_{n-1} & a_4-e_{n-2} & \cdots & a_{n-1}-e_3 & a_n-e_2 \\
\vdots & \vdots & \vdots & \vdots & & & & \ddots
\end{array} \tag{16}$$

Note that (16) is obtained from (12) by, in each row, moving the $-e_1$'s from the last position in the row to the first, and then shifting all other $-e_j$'s in the row one place to the right.

Proof (of Lemma 3.5): Again all the boundary conditions are obvious, so we focus on $m > k \geq 1$. Because $a_1 = e_1$, we have that the sum of the weights of paths from s_m to t_k in (16) is the same as the sum of the weights of paths from s_m to t_k in the following (finite) network:

$$\begin{array}{lllllll} 0 & & & & & & \\ 0 & a_2 - e_2 & & & & & \\ 0 & a_2 - e_3 & a_3 - e_2 & & & & \\ 0 & a_2 - e_4 & a_3 - e_3 & a_4 - e_2 & & & \\ 0 & a_2 - e_5 & a_3 - e_4 & a_4 - e_3 & a_5 - e_2 & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 0 & a_2 - e_m & a_3 - e_{m-1} & a_4 - e_{m-2} & \cdots & a_{m-1} - e_3 & a_m - e_2. \end{array} \tag{17}$$

The proof of Lemma 3.4 shows that this quantity is symmetric in $e_2, \ldots, e_m$, and so the transformation $e_2 \to e_m$, $e_3 \to e_{m-1}$, etc, while maybe changing the path-matrix, does not change the sum of the weights of the paths from s_m to t_k (i.e., the (m,k)-entry of the path matrix). We can also replace the first column of (17) with

$$\begin{array}{c} a_1 - e_m \\ a_1 - e_{m-1} \\ \vdots \\ a_1 - e_2 \\ a_1 - e_1 \end{array}$$

without changing the sum of the weights of the paths from s_m to t_k: the $a_1 - e_1$ at the bottom of the first column (in the $[m,1]$-position) equals 0, which has the effect that the remaining entries in the column do not appear in any paths of non-zero weight. We now have the following network, in which the sum of the weights of the paths from s_m to t_k is as it was in (16):

$$\begin{array}{ccccccc} a_1 - e_m & & & & & & \\ a_1 - e_{m-1} & a_2 - e_m & & & & & \\ a_1 - e_{m-2} & a_2 - e_{m-1} & a_3 - e_m & & & & \\ a_1 - e_{m-3} & a_2 - e_{m-2} & a_3 - e_{m-1} & a_4 - e_m & & & \\ a_1 - e_{m-4} & a_2 - e_{m-3} & a_3 - e_{m-2} & a_4 - e_{m-1} & a_5 - e_m & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ a_1 - e_1 & a_2 - e_2 & a_3 - e_3 & a_4 - e_4 & \cdots & a_{m-1} - e_{m-1} & a_m - e_m. \end{array}$$

The relevant portion of this network, with respect to paths from s_m to t_k (the only vertical edges that can be traversed in a path from s_m to t_k), is exactly (14), and so the proof of Lemma 3.4 shows that the sum of the weights of the paths from s_m to t_k is $S^{\mathbf{a},\mathbf{e}}(m,k)$. $\square$

We refer to the operation that transforms the network (12) of Lemma 3.4 to the network (16) of Lemma 3.5 as *pivoting* on the $[1,1]$-position. More generally, given a planar network of the type shown in Figure 1 in which, for each $m \geq 1$ and $1 \leq k \leq m$ the weight in $[m,k]$-position is of the form $a_{f(m,k)} - e_{g(m,k)}$ (for some functions f, g) we define *pivoting* on the $[m,k]$-position to mean the following:

- the weight in the $[m,k]$-position remains unchanged;
- in row $m+1$, the weights $a_{f(m+1,k)} - e_{g(m+1,k)}$ and $a_{f(m+1,k+1)} - e_{g(m+1,k+1)}$ (in the $[m+1,k]$- and $[m+1,k+1]$-positions, respectively) are replaced with $a_{f(m+1,k)} - e_{g(m+1,k+1)}$ and $a_{f(m+1,k+1)} - e_{g(m+1,k)}$;
- in general, for $\ell \geq 1$ the weights

$$a_{f(m+\ell,k)} - e_{g(m+\ell,k)},\; a_{f(m+\ell,k+1)} - e_{g(m+\ell,k+1)},\; \ldots,\; a_{f(m+\ell,k+\ell)} - e_{g(m+\ell,k+\ell)}$$

  (in the $[m+\ell,k]$- through $[m+\ell,k+\ell]$-positions, respectively) are replaced with

$$a_{f(m+\ell,k)} - e_{g(m+\ell,k+\ell)},\; a_{f(m+\ell,k+\ell)} - e_{g(m+\ell,k)},\; \ldots,\; a_{f(m+\ell,k+\ell)} - e_{g(m+\ell,k+\ell-1)};$$

- and all other weights remain unchanged.

We refer to the triangle consisting of the $[m+\ell_1, k+\ell_2]$-positions for all $\ell_1 \geq 0$ and $0 \leq \ell_2 \leq \ell_1$ as the triangle *headed* at the $[m,k]$-position; Figure 2 shows a portion of the triangle headed at the $[3,2]$-position in a general weighted planar network of the type first introduced in Figure 1.

$$\begin{array}{ccccccc}
x_{11} \\
x_{21} & x_{22} \\
x_{31} & \mathbf{x_{32}} & x_{33} \\
x_{41} & \mathbf{x_{42}} & \mathbf{x_{43}} & x_{44} \\
x_{51} & \mathbf{x_{52}} & \mathbf{x_{53}} & \mathbf{x_{54}} & x_{55} \\
\vdots & \vdots & & & \ddots & \ddots \\
x_{n1} & \mathbf{x_{n2}} & \mathbf{x_{n3}} & \mathbf{x_{n4}} & \cdots & \mathbf{x_{n(n-1)}} & x_{nn} \\
\vdots & \vdots & & & & \ddots & & \ddots
\end{array}$$

Figure 2: The triangle headed at the $[3,2]$-position (in bold)

We can easily generalize Lemma 3.5.

Lemma 3.6. *Let* $\mathbf{a}$ *and* $\mathbf{e}$ *be arbitrary. If a planar network is obtained from (12) of by pivoting on the* $[m,k]$*-position, then as long as the weight in that position is* 0*, the path-matrix of the resulting network is still* $S^{\mathbf{a},\mathbf{e}}$*.*

Furthermore, if $(m_1, m_2, \ldots)$ *and* $(k_1, k_2, \ldots)$ *are sequences satisfying that for each* $i \geq 1$*, the* $[m_{i+1}, k_{i+1}]$*-position is located in the triangle headed at the* $[m_i, k_i]$*-position, and if a planar network is obtained from (12) by first pivoting on the* $[m_1, k_1]$*-position, then pivoting on the* $[m_2, k_2]$*-position of the resulting network, and so on, then as long the weights in each of the positions at which pivoting occurs is* 0*, the path-matrix of the resulting network is still* $S^{\mathbf{a},\mathbf{e}}$*.*

Proof: We begin with the first statement. For a given source s_p and sink t_q, the collection of paths from s_p to t_q can be partitioned according to the first vertex along the path, v_f say, that is part of the triangle headed at the $[m, k]$-position, and the last such vertex, v_l say. By the construction of the network, the portion of the path lying between v_f and v_l lies completely inside the triangle headed at the $[m, k]$-position (note that this portion may be empty, if the path completely avoids the relevant triangle). For a given partition class, the sum of the weights of the paths from s_p to t_q is the product of three terms: the sum of the weights of the paths s_p to v_f, the sum of the weights of the paths v_f to v_l, and sum of the weights of the paths v_l to t_q. The first and third of these sums remain unchanged after pivoting on the $[m, k]$-position, because the pivoting does not change any of the weights away from the triangle headed at the $[m, k]$-position. The middle sum also remains unchanged after pivoting, by Lemma 3.5. Summing over partition classes, the first statement of the lemma follows.

The second statement of the lemma is obtained by iterating the above argument. □

We can now fairly swiftly present the proof of Theorem 1.2.

Proof (of Theorem 1.2): Let $\mathbf{a}$ be non-decreasing. We begin by arguing that if $\mathbf{e}$ is a restricted growth sequence relative to $\mathbf{a}$, then $S^{\mathbf{a},\mathbf{e}}$ is totally non-negative (item 1).

- If all e_i are at most a_1, then the original network (12) presented in Lemma 3.4 has all non-negative weights, and by Lemma 3.3 (Lindström's Lemma) we are done.

- If not, then there is some index j such that $e_j = a_1$ and $e_{j'} < a_1$ for all $j' < j$. We pivot on the $[j, 1]$-position (note that the weight in this position is $a_1 - e_j$, or 0). From the first part of Lemma 3.6 the path-matrix of the resulting network is $S^{\mathbf{a},\mathbf{e}}$. Notice that all weights in the first column of the new network are either positive (the entries in the first $j - 1$ rows) or 0 (the remaining entries), and that all weights in the new network that lie above the triangle headed at the $[j, 1]$-position are positive (they were positive in the original network — here we use that $\mathbf{a}$ is non-decreasing — and remain unchanged after pivoting). In other words, after pivoting all weights in the network in positions outside the triangle headed at the $[j, 1]$-position are non-negative.

- If all e_i for $i > j$ are at most a_2, then the new network has only non-negative weights, and again by Lemma 3.3 we are done. If not, there is some index j' such that $e_{j'} = a_2$ and $e_{j''} < a_2$ for all $j < j'' < j'$. We now pivot on $[j', 2]$-position (which has weight $a_2 - e_{j'} = 0$). Because the $[j', 2]$-position is in the triangle headed at the $[j, 1]$-position, we can apply the second part of Lemma 3.6 to conclude that the path-matrix of the resulting network is still $S^{\mathbf{a},\mathbf{e}}$. Arguing as before, the new weighted planar network has non-negative weights outside the triangle headed at the $[j', 2]$-position.

- Iterating this process (either finitely many times or countably many times, depending on whether $\mathbf{a}$ and $\mathbf{e}$ are finite or countably infinite) we arrive at a weighted planar network all of whose entries are non-negative and whose path-matrix is $S^{\mathbf{a},\mathbf{e}}$; the result now follows from Lemma 3.3.

To complete the proof of Theorem 1.2, we show that if $\mathbf{e}$ is not a restricted growth sequence relative to $\mathbf{a}$, then $S^{\mathbf{a},\mathbf{e}}$ is *not* totally non-negative, and that moreover the failure of total non-negativity is witnessed by a negative matrix entry (item 2).

- Suppose that the failure of $\mathbf{e}$ to be a restricted growth sequence relative to $\mathbf{a}$ is witnessed by some index j such that $e_i < a_1$ for all $i < j$, and $e_j > a_1$. Then evidently the path-matrix associated with (12) has the negative entry $(a_1 - e_j)(a_1 - e_{j-1}) \cdots (a_1 - e_1)$ — it is the $(j, 0)$-entry.

- Otherwise, there is some index j such that $e_i < a_1$ for all $i < j$, and $e_j = a_1$. Consider the network (12), and pivot on the $[j, 1]$-position to obtain a network which, as established in the proof of item 1 above, has path-matrix $S^{\mathbf{a},\mathbf{e}}$. This network has strictly positive weights in the first $j - 1$ entries of the first column, the weights in the rest of the first column are all 0, and all weights above the triangle headed at the $[j, 1]$-position are strictly positive.

  Now suppose that the failure of $\mathbf{e}$ to be a restricted growth sequence relative to $\mathbf{a}$ is witnessed by some index j' such that $e_i < a_2$ for all $j < i < j'$, and $e_{j'} > a_2$. Evidently the $(j', 1)$-entry of the path-matrix is negative, because all paths from $s_{j'}$ to t_1 that do not have weight 0 have a weight which is a product of strictly positive terms, together with the term $a_2 - e_{j'}$, which is negative.

- Continuing in this manner, we find that if the earliest witness of the failure of $\mathbf{e}$ to be a restricted growth sequence relative to $\mathbf{a}$ is some index $\tilde{j}$ with $e_{\tilde{j}} > a_\ell$ for some ℓ, then the $(\tilde{j}, \ell - 1)$-entry of $S^{\mathbf{a},\mathbf{e}}$ is negative.

□

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