

A threshold phenomenon for random independent sets in the discrete hypercube

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Abstract

Let I be an independent set drawn from the discrete d -dimensional hypercube $Q_d = \{0, 1\}^d$ according to the hard-core distribution with parameter $\lambda > 0$ (that is, the distribution in which each set I is chosen with probability proportional to $\lambda^{|I|}$). We show a sharp transition around $\lambda = 1$ in the appearance of I : for $\lambda > 1$, $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = 0$ almost surely, where \mathcal{E} and \mathcal{O} are the bipartition classes of Q_d , whereas for $\lambda < 1$, $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\}$ is almost surely exponential in d .

A key step in the proof is an estimation of $Z_\lambda(Q_d)$, the sum over independent sets in Q_d with each set I given weight $\lambda^{|I|}$ (a.k.a. the hard-core partition function). We obtain the asymptotics of $Z_\lambda(Q_d)$ for $\lambda > \sqrt{2} - 1$, and nearly matching upper and lower bounds for $\lambda \leq \sqrt{2} - 1$, extending work of Korshunov and Sapozhenko.

We also derive a long-range influence result. For all fixed $\lambda > 0$, if I is chosen from the independent sets of Q_d according to the hard-core distribution with parameter λ , conditioned on a particular $v \in \mathcal{E}$ being in I , then the probability that another vertex w is in I is $o(1)$ for $w \in \mathcal{O}$ but $\Omega(1)$ for $w \in \mathcal{E}$.

1 Introduction and statement of results

The focus of this note is the discrete hypercube Q_d . This is the graph on vertex set $V = \{0, 1\}^d$ with two strings adjacent if they differ on exactly

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one coordinate. It is a d -regular bipartite graph with bipartition classes \mathcal{E} and \mathcal{O} , where \mathcal{E} is the set of vertices with an even number of 1's. Note that $|\mathcal{E}| = |\mathcal{O}| = 2^{d-1}$. (For graph theory basics, see e.g. [1]).

An *independent set* in Q_d is a set of vertices no two of which are adjacent. Write $\mathcal{I}(Q_d)$ for the set of independent sets in Q_d . The *hard-core model* with parameter λ on Q_d (abbreviated $\text{hc}(\lambda)$) is the probability distribution on $\mathcal{I}(Q_d)$ in which each I is chosen with probability proportional to $\lambda^{|I|}$. This fundamental statistical physics model interprets vertices of Q_d (or other graph) as sites that may or may not be occupied by massive particles, and edges as bonds between pairs of sites (encoding, for example, spatial proximity). The occupation rule is that bonded sites may not be simultaneously occupied, so a legal configuration of particles corresponds to an independent set in the graph. In this context λ represents a density parameter, with larger λ favouring denser configurations. (For an introduction to the hard-core model from a combinatorial perspective, see for example [2].)

In [8], entropy methods are used to make an extensive study of the hard-core model on Q_d (and regular bipartite graphs in general) for fixed $\lambda > 0$. One of the main results is that an independent set from Q_d chosen according to $\text{hc}(\lambda)$ exhibits *phase coexistence* — it comes either predominantly from \mathcal{E} or predominantly from \mathcal{O} . Specifically, it is shown in [8] that for fixed $\lambda, \varepsilon > 0$, and for I chosen from $\mathcal{I}(Q_d)$ according to $\text{hc}(\lambda)$, both of

$$\begin{aligned} \left| |I| - \frac{\lambda}{1+\lambda} 2^{d-1} \right| &\leq \frac{2^d}{d^{1-\varepsilon}}, \\ \min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} &\leq \frac{2^d}{d^{1/2-\varepsilon}} \end{aligned}$$

hold almost surely (with probability tending to 1 as $d \rightarrow \infty$). Informally, the work of [8] demonstrates that for all fixed $\lambda > 0$, $\text{hc}(\lambda)$ is close to $\frac{1}{2}\mu_{\mathcal{E}} + \frac{1}{2}\mu_{\mathcal{O}}$ where $\mu_{\mathcal{E}}$ (or $\mu_{\mathcal{O}}$) is a random subset of \mathcal{E} (or \mathcal{O}) in which each vertex is chosen to be in the set independently with probability $\frac{\lambda}{1+\lambda}$. (This is just $\text{hc}(\lambda)$ on \mathcal{E} (or \mathcal{O}).)

Kahn's estimates on $|I|$ and $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\}$ do not involve λ . Here we are able to obtain more precise estimates that capture the dependence on λ and in particular show that $\lambda = 1$ is a critical value around which a transition occurs in the nature of the phase coexistence: for $\lambda > 1$, the smaller of $|I \cap \mathcal{E}|, |I \cap \mathcal{O}|$ is almost surely 0, whereas for $\lambda < 1$, it is almost surely exponential in d . Allowing λ to vary with d , we find that the transition

between $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\}$ being almost surely 0 and almost surely going to infinity with d occurs in an interval of length order $1/d$.

To state our results precisely we consider four possible ranges of λ :

$$\lambda \geq 1 + \frac{\omega(1)}{d} \quad (1)$$

$$|\lambda - 1| \leq \frac{O(1)}{d} \quad (2)$$

$$\sqrt{2} - 1 + \frac{(\sqrt{2} + \Omega(1)) \log d}{d} \leq \lambda \leq 1 - \frac{\omega(1)}{d} \quad (3)$$

$$\frac{c \log d}{d^{1/3}} \leq \lambda \leq \sqrt{2} - 1 + \frac{(\sqrt{2} + o(1)) \log d}{d} \quad (4)$$

where $c > 0$ is an absolute constant (that we do not explicitly compute). Here and in what follows, $\omega(1)$ indicates a function of d that tends to infinity as d does; $o(1)$ a function that tends to 0; $\Omega(1)$ a function that is eventually always greater than some constant greater than 0; and $O(1)$ a function that is bounded above by a constant. All implied constants will be independent of d , all limiting statements are as $d \rightarrow \infty$, and where we are not taking a limit, we will always assume that d is large enough to support our assertions. Unless otherwise indicated, all logarithms are base e .

Theorem 1.1 *Let I be chosen from $\mathcal{I}(Q_d)$ according to $\text{hc}(\lambda)$.*

1. *For λ satisfying (1), almost surely*

$$\left| \max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} - \frac{\lambda 2^{d-1}}{1 + \lambda} \right| \leq 2^{d/2} \sqrt{\log d} \quad (5)$$

and

$$\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = 0. \quad (6)$$

2. *For λ satisfying (2), almost surely (5) holds. If $\lambda \sim 1 + \frac{k}{d}$ for some constant k then for each $c \in \mathbb{N}$*

$$P(\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = c) \sim \frac{\left(\frac{1}{2}e^{-k/2}\right)^c}{c!} \exp\left\{-\frac{1}{2}e^{-k/2}\right\}. \quad (7)$$

3. *For λ satisfying (3), almost surely (5) holds, as well as*

$$\frac{\left| \min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} - \frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d \right|}{\sqrt{(2 + \varepsilon) \frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d \log \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d\right)}} \leq 1 \quad (8)$$

where $\varepsilon > 0$ is arbitrary.

4. For λ satisfying (4), almost surely

$$\left| \max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} - \frac{\lambda 2^{d-1}}{1 + \lambda} \right| \leq d \log d \left(\frac{2}{1 + \lambda} \right)^d \quad (9)$$

and

$$\frac{\lambda}{8 \log m} \left(\frac{2}{1 + \lambda} \right)^d \leq \min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq \frac{em^2 \lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d \quad (10)$$

where $m = m(\lambda, d) < d/10$ is any natural number satisfying

$$(ed^2)^m \lambda^{m+1} (1 + \lambda)^{2m(m+1)} \frac{2^d}{(1 + \lambda)^{d(m+1)}} = o(1). \quad (11)$$

In particular, $P(\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = 0)$ goes from $1 - o(1)$ to $o(1)$ as λ goes from $1 + \omega(1/d)$ to $1 - \omega(1/d)$. Also, for fixed $\lambda \leq \sqrt{2} - 1$ we satisfy (11) by taking $m = \lceil 1/\log_2(1 + \lambda) \rceil$ and so combining (8) and (10) we find that for all fixed $\lambda < 1$, there are constants $c(\lambda)$ and $C(\lambda)$ (independent of d) such that almost surely

$$c(\lambda) \left(\frac{2}{1 + \lambda} \right)^d \leq \min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq C(\lambda) \left(\frac{2}{1 + \lambda} \right)^d.$$

To understand probabilities associated with the hard-core model, it is useful to understand the normalizing constant (or partition function)

$$Z_\lambda(Q_d) = \sum_{I \in \mathcal{I}(Q_d)} \lambda^{|I|}.$$

In the case $\lambda = 1$, this just counts the number of independent sets in Q_d . Motivated by the interpretation of independent sets as “codes of distance 2” over a binary alphabet, Korshunov and Sapozhenko [9] gave an asymptotic estimate in this case.

Theorem 1.2 $|\mathcal{I}(Q_d)| \sim 2\sqrt{e}2^{2^{d-1}}$ as $d \rightarrow \infty$.

The following theorem, which extends Theorem 1.2 to a wider range of λ , is the main tool in our approach to Theorem 1.1.

Theorem 1.3

$$Z_\lambda(Q_d) = \begin{cases} (2 + o(1))(1 + \lambda)^{2^{d-1}} & \text{if } \lambda \text{ satisfies (1)} \\ (2 + o(1))(1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right\} & \text{if } \lambda \text{ satisfies (2) or (3)} \\ (1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d (1 + o(1)) \right\} & \text{if } \lambda \text{ satisfies (4)}. \end{cases}$$

Remark 1.4 *Very few specific properties of Q_d are used in the sequel. We just use the fact that it is a regular bipartite graph which satisfies certain isoperimetric bounds (specifically, those of Lemmas 3.5 and 3.6). Analogues of Theorems 1.1 and 1.3 could be obtained for any family of regular bipartite graphs with appropriate isometric properties. In the absence of an appealing general formulation, we confine ourselves here to considering Q_d .*

The phenomenon of phase coexistence is, not unsurprisingly, accompanied by a long-range influence phenomenon. An independent set I chosen from $\mathcal{I}(Q_d)$ according to $\text{hc}(\lambda)$ is drawn (in the range of λ that we are considering) either predominantly from \mathcal{E} or predominantly from \mathcal{O} . If we are given the information that a particular vertex ($v \in \mathcal{E}$, say) is in I , then that should make it very likely that I is drawn mostly from \mathcal{E} . So if we then ask what is the probability that another vertex (w , say) is in I , the answer should depend on the parity of w , being quite small if $w \in \mathcal{O}$ and reasonably large if $w \in \mathcal{E}$. This heuristic can be made rigorous.

Theorem 1.5 *Let λ satisfy $\lambda > \frac{c \log d}{d^{1/3}}$. Let I be chosen from $\mathcal{I}(Q_d)$ according to $\text{hc}(\lambda)$. If $u_d, v_d \in \mathcal{E}$ and $w_d \in \mathcal{O}$ are three (sequences of) vertices in Q_d then*

$$P(u_d \in I \mid w_d \in I) \leq (1 + \lambda)^{-d(1-o(1))} \quad (12)$$

and

$$P(u_d \in I \mid v_d \in I) \geq \frac{\lambda}{1 + \lambda} (1 - o(1)). \quad (13)$$

Estimates of $Z_\lambda(Q_d)$ can also be used to obtain information on the number of independent sets of Q_d of a given size; this topic will be explored in detail in a subsequent paper [4].

An overview of our approach is given in Section 2. The main technical lemma (Lemma 3.7) is stated and proved in Section 3, along with notation and other useful lemmas. The proofs of all the stated theorems appear in Section 4.

2 Overview

A trivial lower bound on $Z_\lambda(Q_d)$ for all $\lambda > 0$ is $2(1+\lambda)^{2^{d-1}} - 1$: just consider the contribution from those sets which are drawn either entirely from \mathcal{E} or entirely from \mathcal{O} . To improve this to the lower bounds appearing in Theorem 1.3, we consider not just independent sets which are confined purely to either \mathcal{E} or \mathcal{O} . It is easy to see that there is a contribution of

$$2^{d-1}\lambda(1+\lambda)^{2^{d-1}-d} = (1+\lambda)^{2^{d-1}} \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d$$

from those independent sets that have just one vertex from \mathcal{O} , (and the same from those that have just one vertex from \mathcal{E}) and more generally a contribution of approximately

$$2(1+\lambda)^{2^{d-1}} \frac{1}{k!} \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right)^k$$

from those independent sets which consist of exactly k non-nearby vertices on one side of the bipartition, for reasonably small k (by “non-nearby” it is meant that there are no common neighbours between pairs of the vertices). Indeed, there are 2 ways to chose the bipartition class that has k vertices, and approximately $\binom{2^{d-1}}{k} \approx \frac{1}{k!} 2^{(d-1)k}$ ways to choose the k vertices. These vertices together have a neighbourhood of size kd , so the sum of the weights of independent sets that extend the k vertices is $\lambda^k (1+\lambda)^{2^{d-1}-kd}$.

Summing over k we get a lower bound on $Z_\lambda(Q_d)$ of approximately

$$2(1+\lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right\}.$$

This lower bound could have also been achieved by summing only from k a little below to a little above $\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d$ (where the mass of the Taylor series of $\exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right\}$ is concentrated) and, once k vertices have been chosen from one side, only considering extensions to the other side which have close to $\frac{\lambda 2^{d-1}}{1+\lambda}$ vertices (where the mass of the binomial series $(1+\lambda)^{2^{d-1}}$ is concentrated). This does not cause the count of extensions to drop much below $(1+\lambda)^{2^{d-1}-dk}$ as long as dk is much smaller than 2^{d-1} , which it will be for $k \approx \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d$ and $\lambda > \frac{c \log d}{d^{1/3}}$. In this way we see that the lower

bound on $Z_\lambda(Q_d)$ can be achieved by only considering independent sets I with $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \approx \frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d$ and $\max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \approx \frac{\lambda 2^{d-1}}{1+\lambda}$. Thus an upper bound that nearly matches the lower bound completes the proofs of both Theorems 1.3 and 1.1.

To motivate the upper bound, consider what happens when we count the contribution from independent sets that have exactly two nearby vertices from \mathcal{O} (that is, two vertices with a common neighbour). There are approximately $d^2 2^d$ choices for this pair (as opposed to approximately 2^{2d} choices for a pair of vertices without a common neighbour), since once the first vertex has been chosen the second must come from the approximately d^2 vertices at distance two from the first. The sum of the weights of independent sets that extend each choice is $\lambda^2(1+\lambda)^{2^{d-1}-2d+2}$, roughly the same as the sum of the weights of extensions in the case of the pair of vertices without a common neighbour. The key point here is that any pair of vertices from \mathcal{O} has at most two neighbours in common, so has neighbourhood size of approximately $2d$, whether or not the vertices are nearby. Thus we get an additional contribution of approximately

$$(1+\lambda)^{2^{d-1}} \frac{d^2 2^d}{(1+\lambda)^{2d}}$$

to the partition function from those sets with two nearby vertices from \mathcal{O} , negligible compared to the addition contribution to the partition function from those sets with two non-nearby vertices from \mathcal{O} .

The main work in upper bounding $Z_\lambda(Q_d)$ involves extending this to the observation that the only non-negligible contribution to the partition function comes from independent sets that on one side consist of a set of vertices with non-overlapping neighbourhoods. This in turn amounts to showing that there is a negligible contribution from those independent sets which are “2-linked” on one side (that is, are such that between any two vertices on one side, there is a path in the cube every second vertex of which passes through the independent set.) This entails proving a technical lemma (Lemma 3.7) bounding the sum of the weights of 2-linked subsets of \mathcal{E} of a given size whose neighbourhood in \mathcal{O} is of a given size. This lemma is a weighted generalization of an enumeration result originally introduced by Sapozhenko in [10] and used in [11] to simplify the original proof of Theorem 1.2. A weaker form of Lemma 3.7 is proved in [6] where it used to estimate the weighted sum of independent sets in Q_d satisfying $|I \cap \mathcal{E}| = |I \cap \mathcal{O}|$.

3 Preliminaries

For $A \subseteq V (= \{0, 1\}^d)$ write $N(A)$ for the set of vertices outside V that are neighbours of a vertex in A , and set

$$[A] = \{v \in V : N(\{v\}) \subseteq N(A)\}$$

Say that $A \subseteq \mathcal{E}$ (or \mathcal{O}) is *small* if $|[A]| \leq 2^{d-2}$ and *2-linked* if $A \cup N(A)$ induces a connected subgraph of Q_d . Note that if A is 2-linked then so is $[A]$. Any A can be decomposed into its maximal 2-linked subsets; we refer to these as the *2-components* of A .

The following lemma bounds the number of connected subsets of a graph; see [5, Lemma 2.1]. (The bound given in [5] is $(e\Delta)^n$, but the proof easily gives the claimed improvement.)

Lemma 3.1 *Let G be a graph on vertex set $V(G)$ with maximum degree Δ . The number of n -vertex subsets of $V(G)$ which contain a fixed vertex and induce a connected subgraph is at most $(e\Delta)^{n-1}$.*

The following is a special case of Hoeffding's Inequality [7].

Lemma 3.2 *For all $\lambda > 0$, $\delta > 0$ and $m \in \mathbb{N}$,*

$$\sum_{j=\lfloor m(\frac{\lambda}{1+\lambda}-\delta) \rfloor}^{\lceil m(\frac{\lambda}{1+\lambda}+\delta) \rceil} \lambda^j \binom{m}{j} \geq (1 - 2 \exp\{-2\delta^2 m\}) (1 + \lambda)^m.$$

We will need to compare the exponential function e^x to truncates $e_D(x) = \sum_{k=0}^D \frac{x^k}{k!}$ of its Taylor series; the following will be sufficient.

Lemma 3.3 *For any $y \leq x < z$ with $y, z \in \mathbb{N}$,*

$$e_y(x) \leq \exp \left\{ y \log \frac{ex}{y} + \log(y + 1) \right\} \quad (14)$$

and

$$e^x - e_z(x) \leq \exp \left\{ z \log \frac{ex}{z} + \log \left(\frac{x}{z-x} \right) \right\} \quad (15)$$

Proof: We have

$$e_y(x) = \sum_{k=0}^y \frac{x^k}{k!} \leq (y+1) \frac{x^y}{y!} \leq \exp \left\{ y \log \left(\frac{ex}{y} \right) + \log(y+1) \right\}$$

and

$$e^x - e_z(x) = \sum_{k=z+1}^{\infty} \frac{x^k}{k!} \leq \frac{x^z}{z!} \sum_{k=1}^{\infty} \left(\frac{x}{z} \right)^k \leq \exp \left\{ z \log \left(\frac{ex}{z} \right) + \log \left(\frac{x}{z-x} \right) \right\}$$

in both cases using $n! \geq (n/e)^n$. \square

Corollary 3.4 *Let $\{x_d\}_{d=1}^{\infty}$ be such that $x_d \rightarrow \infty$. With $\varepsilon_1 = \sqrt{c_1 \log x_d/x_d}$ and $\varepsilon_2 = \sqrt{c_2 \log x_d/x_d}$ where $c_1 > 2$ and $c_2 > 1$ are constants, we have*

$$e_{[(1+\varepsilon_2)x_d]}(x_d) - e_{[(1-\varepsilon_1)x_d]}(x_d) \sim e^{x_d}.$$

Proof: Note that the function $f(t) = (1+t) \log \left(\frac{e}{1+t} \right)$ has a local maximum at $t = 0$ and for $t = o(1)$ satisfies $f(t) = 1 - \frac{t^2}{2} + o(t^2)$. From (14) we have $e_{[(1-\varepsilon_1)x_d]}(x_d) = o(e^{x_d})$ and from (15) we have $e^{x_d} - e_{[(1+\varepsilon_2)x_d]}(x_d) = o(e^{x_d})$. \square

We will need the following isoperimetric bounds for $A \subseteq \mathcal{E}$ (or \mathcal{O}) (see [3, Lemma 6.2] for the first and [9, Lemma 1.3] for the second).

Lemma 3.5 *There is a constant $C > 0$ such that for $A \subseteq \mathcal{E}$ (or \mathcal{O}), if $|A| \leq d^4$ then $|A| \leq C|N(A)|/d$. If $|A| \leq d/10$, then $|N(A)| \geq d|A| - 2|A|(|A| - 1)$.*

Lemma 3.6 *For $A \subseteq \mathcal{E}$ (or \mathcal{O}), if $|A| \leq 2^{d-2}$ then*

$$|N(A)| \geq \left(1 + \Omega(1/\sqrt{d}) \right) |A|.$$

Our main tool is a weighted version of the main result of [10].

Lemma 3.7 *For each $a, g \geq 1$, set*

$$\mathcal{G}(a, g) = \{A \subseteq \mathcal{E} \text{ 2-linked} : |[A]| = a \text{ and } |N(A)| = g\}.$$

There are constants $c > 0$ and $c' > 0$ such that the following holds. If $\lambda > \frac{c \log d}{d^{1/3}}$ then

$$\sum_{A \in \mathcal{G}(a, g)} \lambda^{|A|} \leq 2^d (1 + \lambda)^g \exp \left\{ -\frac{c'(g-a) \log d}{d^{2/3}} \right\}.$$

Before giving the proof, we establish the following corollary, which is all that we will use in the sequel.

Corollary 3.8 For $\lambda > \frac{c \log d}{d^{1/3}}$ and $m \leq d/10$,

$$\sum \lambda^{|A|} (1 + \lambda)^{-|N(A)|} \leq (ed^2)^{m-1} \lambda^m (1 + \lambda)^{2m(m-1)} \frac{2^d}{(1 + \lambda)^{md}}.$$

where the sum is over all $A \subseteq \mathcal{E}$ small and 2-linked with $|A| \geq m$.

Proof: We consider the sum in three parts. Say that A is of *type I* if $|A| \leq d/10$; of *type II* if $d/10 < |A| \leq d^4$ and of *type III* if $d^4 < |A|$.

For type I A with $|A| = k$ ($k \geq m$) there are (by Lemma 3.1) at most $2^{d-1} (ed^2)^{k-1}$ choices for A (the factor of 2^{d-1} accounting for the choice of a fixed vertex in A and the d^2 coming from the fact that each A is connected in a graph with maximum degree at most d^2). By Lemma 3.5 each such A satisfies $|N(A)| \geq dk - 2k(k-1)$. It follows that the contribution to the sum from type I A 's is at most

$$\sum_{k \geq m}^{d/10} 2^{d-1} (ed^2)^{k-1} \lambda^k (1 + \lambda)^{-dk + 2k(k-1)}.$$

For large enough d (independent of λ) each summand above is at most one third its predecessor and so the total sum is at most

$$\frac{3}{4} (ed^2)^{m-1} \lambda^m (1 + \lambda)^{2m(m-1)} \frac{2^d}{(1 + \lambda)^{md}}.$$

The contribution to the sum from type II A 's (again using Lemmas 3.1 and 3.5) is at most

$$\sum_{k=d/10}^{d^4} 2^{d+3k \log d + k \log \lambda - \frac{1}{C} dk \log(1+\lambda)}$$

(where C is the constant guaranteed by Lemma 3.5). For all λ , this sum is $2^{-\omega(d)}$.

In the range $|A| > d^4$ we partition the possible A 's according to $a := |[A]| > d^4$ and $g := |N(A)| > d^4$. By Lemma 3.7, the sum over type III A 's is at most

$$\sum_{a,g>d^4, \mathcal{G}(a,g)\neq\emptyset} \left(\sum_{A\in\mathcal{G}(a,g)} \lambda^{|A|} \right) (1+\lambda)^{-g} \leq \sum_{a,g>d^4} 2^d \exp \left\{ -\frac{c'(g-a)\log d}{d^{2/3}} \right\}.$$

By Lemma 3.6, $g-a \geq \Omega(d^{7/2})$ and there are at most 2^{2d} choices for a and g , so the sum is at most $2^{-\omega(d)}$. \square

Proof of Lemma 3.7: We begin by recalling a result from [6] which is very similar to Lemma 3.7. A combination of Lemmas 3.2 through 3.4 of that reference yields

$$\begin{aligned} \sum_{A\in\mathcal{G}(a,g)} \lambda^{|A|} &\leq \binom{2^{d-1}}{\leq \frac{2g\log d}{d}} \binom{2g\log d}{\leq \frac{2g}{d}} \binom{2d^3g\log d}{\leq \frac{2t}{\psi}} \binom{2g\log d}{\leq \frac{td}{(d-\psi)\psi}} \times \\ &\quad \max \left\{ (1+\lambda)^{g-\gamma t}, \binom{3dg}{\leq \frac{2t\psi}{d-\psi} + \gamma t} (1+\lambda)^{g-t} \right\}, \end{aligned} \quad (16)$$

for all $a \geq 1$, $g \geq 1$, $1 \leq \psi \leq d/2$, $1 \geq \gamma > \frac{-2\psi}{d-\psi}$ and $\lambda > 0$, where $\binom{n}{\leq r}$ is shorthand for $\sum_{i\leq r} \binom{n}{i}$ and $t = g - a$. The aim of this section is to explain how we can improve the bound in (16) by replacing the first two terms on the right-hand side (coming from Lemmas 3.2 and 3.3 of [6]) with

$$2^d \exp \left\{ O \left(\frac{t \log^7 d}{d} \right) \right\}.$$

First we show that this implies Lemma 3.7. We take $\psi = d^{2/3}$ and

$$\gamma = \frac{\log(1+\lambda) - \frac{6\psi\log d}{d-\psi}}{\log(1+\lambda) + 3\log d} \geq \frac{\frac{c}{3} - 3}{d^{1/3}},$$

with the inequality valid for $\lambda > \frac{c\log d}{d^{1/3}}$. Noting that

$$\binom{3dg}{\leq \frac{2t\psi}{d-\psi} + \gamma t} \leq \exp \left\{ 3\log d \left(\frac{2t\psi}{d-\psi} + \gamma t \right) \right\} = (1+\lambda)^{(1-\gamma)t}$$

(the inequality coming from the simple binomial estimate

$$\binom{n}{\leq k} \leq \exp \left\{ (1+o(1)) \left(k \log \frac{n}{k} \right) \right\} \quad (17)$$

for $k = o(n)$ and the fact that $t \geq \Omega(g/\sqrt{d})$ (see Lemma 3.6), and the equality coming from our choice of γ), we see that

$$\max \left\{ (1 + \lambda)^{g - \gamma t}, \left(\leq \frac{3dg}{\frac{2t\psi}{d - \psi} + \gamma t} \right) (1 + \lambda)^{g - t} \right\} \leq (1 + \lambda)^{g - \gamma t}.$$

Using (17) and $t \geq \Omega(g/\sqrt{d})$, the improved right-hand side of (16) is now seen to be at most

$$2^d (1 + \lambda)^g \exp \left\{ -\gamma t \log(1 + \lambda) + O \left(\frac{t \log d}{d^{2/3}} \right) \right\} \leq 2^d (1 + \lambda)^g \exp \left\{ -\frac{c't \log d}{d^{2/3}} \right\}$$

for some $c' > 0$ (for suitably large $c > 0$), as claimed.

To facilitate the explanation of how we improve (16), it is helpful to summarize the conclusions of Lemmas 3.2 through 3.4 of [6] as they apply to Q_d . Lemma 3.2 asserts that for all a and g , there is a family $\mathcal{V} = \mathcal{V}(a, g) \subseteq 2^{\mathcal{O}}$ with $|\mathcal{V}| \leq \binom{2^{d-1}}{2g \log d/d}$ such that for each $A \in \mathcal{G}(a, g)$ there is $F_0 = F_0(A) \in \mathcal{V}$ satisfying $F_0 \subseteq N(A)$, $N(F_0) \supseteq [A]$ and $|N(A) \setminus F_0| \leq g$. Lemma 3.3 asserts that for each $a, g, F_0 \in \mathcal{V}(a, g)$ and $1 \leq \psi \leq d/2$, there is a family $\mathcal{W} = \mathcal{W}(F_0, \psi, a, g) \subseteq 2^{\mathcal{O}} \times 2^{\mathcal{E}}$ with $|\mathcal{W}| \leq \binom{2g \log d}{\leq 2g/d} \binom{2d^3 g \log d}{\leq 2t/\psi} \binom{2g \log d}{\leq td/(\psi(d-\psi))}$ such that for any $A \in \mathcal{G}(a, g)$ for which F_0 satisfies the conclusion of Lemma 3.2, there is a pair $(F, S) \in \mathcal{W}$ such that $F \subseteq N(A)$, $S \supseteq [A]$ and $|S| \leq |F| + 2t\psi/(d - \psi)$. Finally, Lemma 3.4 asserts that for each $a, g, 1 \leq \psi \leq d/2, 1 \geq \gamma > -2\psi/(d - \psi)$ and $\lambda > 0$, for each $(F, S) \in 2^{\mathcal{O}} \times 2^{\mathcal{E}}$ that satisfies $|S| \leq |F| + 2t\psi/(d - \psi)$ we have

$$\sum \lambda^{|A|} \leq \max \left\{ (1 + \lambda)^{g - \gamma t}, \left(\leq \frac{3dg}{\frac{2t\psi}{d - \psi} + \gamma t} \right) (1 + \lambda)^{g - t} \right\}$$

where the sum is over all those $A \in \mathcal{G}(a, g)$ satisfying $F \subseteq N(A)$ and $S \supseteq [A]$. Combining these three lemmas we get (16).

To obtain the claimed improvement over (16), we do two things. Firstly, we replace Lemma 3.2 of [6] with the following lemma of Sapozhenko [10, Lemma 4.5].

Lemma 3.9 *There is a constant $c' > 0$ such that the following holds. Let G be a d -regular bipartite graph with bipartition classes X and Y and let a and g be such that $t (= g - a) > \frac{\log_2^3 d}{d^2}$. Set*

$$\mathcal{H}(a, g) = \{A \subseteq X \text{ small} : |[A]| = a \text{ and } |N(A)| = g\}.$$

There is a family $\mathcal{V} = \mathcal{V}(a, g) \subseteq 2^Y$ with

$$|\mathcal{V}| \leq |X| \exp \left\{ \frac{c' \log^7 d}{d} \left(\frac{g}{d} + t \right) \right\} \quad (18)$$

such that for each $A \in \mathcal{H}(a, g)$ there is $F_0 = F_0(A) \in \mathcal{V}$ satisfying $F_0 \subseteq N(A)$, $N(F_0) \supseteq [A]$ and $|N(A) \setminus F_0| \leq 2t$.

The properties of F_0 established in [10, Lemma 4.5] are that $N(A)' \subseteq F_0 \subseteq N(A)$ and $N(F_0) \supseteq [A]$, where

$$N(A)' = \{v \in N(A) : |N(\{v\}) \cap [A]| > d/\log^5 d\}.$$

From this it easily follows that $|N(A) \setminus F_0| \leq 2t$. Indeed, there are td edges going from $N(A)$ to $X \setminus [A]$, and each vertex in $N(A) \setminus F_0$ is in $N(A) \setminus N(A)'$ and so contributes at least $d(1 - \log^{-5} d) > d/2$ distinct edges to this set.

Specializing Lemma 3.7 to Q_d (where, by Lemma 3.6, $t \geq \Omega(g/\sqrt{d})$, so the first term in the exponent of (18) is negligible compared to the second), the bound on \mathcal{V} becomes

$$|\mathcal{V}| \leq 2^d \exp \left\{ O \left(\frac{t \log^7 d}{d} \right) \right\}. \quad (19)$$

This replaces the first term in (16). Secondly, we make one small modification to [6, Lemma 3.3]. From the proof of the lemma, we find that the initial term $\binom{2g \log d}{\leq 2g/d}$ can be replaced by $\binom{|N(F_0)|}{\leq 2|N(A) \setminus F_0|/d}$ (for which $\binom{2g \log d}{\leq 2g/d}$ is an upper bound). Since we are replacing Lemma 3.2 of [6] with Lemma 3.9 above, we can bound

$$\binom{|N(F_0)|}{\leq \frac{2|N(A) \setminus F_0|}{d}} \leq \binom{dg}{\leq \frac{4t}{d}} \leq \exp \left\{ O \left(\frac{t \log d}{d} \right) \right\}. \quad (20)$$

This replaces the second term of (16). The remainder of the proof of [6, Lemma 3.3] goes through unchanged, as does [6, Lemma 3.4]. So combining (19) with (20), we obtain the claimed improvement over (16). \square

4 Proofs of the main theorems

4.1 Proof of Theorem 1.3

We will begin with a general upper bound on $Z_\lambda(Q_d)$.

Lemma 4.1 For any $\lambda > 0$,

$$Z_\lambda(Q_d) \leq 2(1 + \lambda)^{2^{d-1}} \exp \left\{ \sum_{A \subseteq \mathcal{E} \text{ small, 2-linked, } |A| \geq 1} \lambda^{|A|} (1 + \lambda)^{-|N(A)|} \right\}.$$

To see that this implies the claimed upper bounds, note that

$$\sum \lambda^{|A|} (1 + \lambda)^{-|N(A)|} = \frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d + \frac{\lambda^2 (1 + \lambda)^2}{4} \binom{d}{2} \frac{2^d}{(1 + \lambda)^{2d}} \quad (21)$$

where the sum is over all $A \subseteq \mathcal{E}$ small and 2-linked with $1 \leq |A| \leq 2$. The second term on the right corresponds to $|A| = 2$: there are $2^{d-1} \binom{d}{2} / 2$ ways to choose $A \subseteq \mathcal{E}$ small and 2-linked with $|A| = 2$, and each such A has $|N(A)| = 2d - 2$. The first term corresponds to $|A| = 1$. On the other hand, from Corollary 3.8 we have that for all $\lambda > \frac{c \log d}{d^{1/3}}$

$$\begin{aligned} \sum \lambda^{|A|} (1 + \lambda)^{-|N(A)|} &\leq (ed^2)^2 \lambda^3 (1 + \lambda)^{12} \frac{2^d}{(1 + \lambda)^{3d}} \\ &= o \left(\frac{\lambda^2 (1 + \lambda)^2}{4} \binom{d}{2} \frac{2^d}{(1 + \lambda)^{2d}} \right) \end{aligned} \quad (22)$$

where the sum is now over all $A \subseteq \mathcal{E}$ small and 2-linked with $|A| \geq 3$. Inserting (21) and (22) into Lemma 4.1 we obtain (for $\lambda > \frac{c \log d}{d^{1/3}}$)

$$Z_\lambda(Q_d) \leq 2(1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d + \lambda^2 (1 + \lambda)^2 d^2 \frac{2^d}{(1 + \lambda)^{2d}} \right\}. \quad (23)$$

If $\lambda = \lambda(d)$ satisfies (1) then the exponent in (23) is $o(1)$. If λ satisfies either (2) or (3) then it is $\frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d + o(1)$. Finally, if λ satisfies (4) then it is $\frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d (1 + o(1))$. This gives all the upper bounds of Theorem 1.3.

Remark 4.2 We expect the bound

$$Z_\lambda(Q_d) \leq (1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d (1 + o(1)) \right\}$$

to be valid for $\lambda > (1 + \Omega(1)) \log d / d$, and this would follow immediately from the extension of Lemma 3.7 to this range of λ . The bound is no longer valid for $\lambda \leq \log d / d$; in this range the contribution to the sum in the exponent in Lemma 4.1 from $|A| = 2$ exceeds the contribution from $|A| = 1$.

Proof of Lemma 4.1: A simple argument (based on the fact that Q_d has a perfect matching) shows that for $I \in \mathcal{I}(Q_d)$, at least one of $||[I \cap \mathcal{E}]| \leq 2^{d-2}$, $||[I \cap \mathcal{O}]| \leq 2^{d-2}$ holds. By \mathcal{E} - \mathcal{O} symmetry we therefore have

$$Z_\lambda(Q_d) \leq 2(1 + \lambda)^{2^{d-1}} \sum_{A \subseteq \mathcal{E} \text{ small}} \lambda^{|A|} (1 + \lambda)^{-|N(A)|}. \quad (24)$$

Decomposing A into 2-components A_1, \dots, A_k , we have

$$\lambda^{|A|} (1 + \lambda)^{-|N(A)|} = \prod_{i=1}^k \lambda^{|A_i|} (1 + \lambda)^{-|N(A_i)|} \quad (25)$$

and

$$\begin{aligned} \sum_{A \subseteq \mathcal{E} \text{ small}} \lambda^{|A|} (1 + \lambda)^{-|N(A)|} &= \sum \left\{ \prod_{i=1}^k \lambda^{|A_i|} (1 + \lambda)^{-|N(A_i)|} : \begin{array}{l} k \geq 0 \\ A \subseteq \mathcal{E} \text{ small} \\ A = \cup_{i=1}^k A_i \end{array} \right\} \\ &\leq \sum_{k \geq 0} \frac{(\sum \lambda^{|A|} (1 + \lambda)^{-|N(A)|})^k}{k!} \\ &= \exp \left\{ \sum \lambda^{|A|} (1 + \lambda)^{-|N(A)|} \right\} \end{aligned} \quad (26)$$

where the unqualified sum in the last two lines is over all $A \subseteq \mathcal{E}$ small and 2-linked with $|A| \geq 1$. Combining (26) with (24) we obtain the lemma. \square

Before turning to the lower bounds, we combine (21), (21) and (26) to observe that for $\lambda > \frac{c \log d}{d^{1/3}}$ (for suitably large c) we have

$$\sum_{A \subseteq \mathcal{E} \text{ small}} \lambda^{|A|} (1 + \lambda)^{-|N(A)|} \leq \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d + \frac{d^2 \lambda^2 (1 + \lambda)^2 2^d}{(1 + \lambda)^{2d}} \right\}. \quad (27)$$

Now we turn to the lower bounds on $Z_\lambda(Q_d)$, which will follow from a general bound that is more than what we need for the proof of Theorem 1.3 but just what we need for much of Theorem 1.1.

Lemma 4.3 *For all $\lambda \geq \frac{\omega(1)}{d}$ and $f \leq \ell \leq \frac{2^{d-2}}{d^2}$,*

$$Z_\lambda(Q_d) \geq 2(1 + \lambda)^{2^{d-1}} \sum_{k=f}^{\ell} \frac{1}{k!} \left(\frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d \right)^k \exp \left\{ -\frac{\ell^2 d^2}{2^{d-2}} \right\} \left(1 - \frac{2}{d^2} \right)$$

This lower bound is obtained by considering only I which satisfy

$$f \leq \min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq \ell$$

and

$$e_1 \leq \max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq e_2$$

where

$$e_1 = \frac{\lambda}{1+\lambda} (2^{d-1} - d\ell) - \sqrt{\log d (2^{d-1} - d\ell)}$$

and

$$e_2 = \frac{\lambda}{1+\lambda} (2^{d-1} - d\ell) + \sqrt{\log d (2^{d-1} - d\ell)}$$

Before proving the lemma, we use it to obtain the claimed lower bounds on $Z_\lambda(Q_d)$ and complete the proof of Theorem 1.3.

For λ satisfying either (1) or (2), $\frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d = O(1)$. With $f = f(\lambda) = 0$ and $\ell = \ell(\lambda) = \log d$ (say), an application of (15) yields

$$\sum_{k=f(\lambda)}^{\ell(\lambda)} \frac{1}{k!} \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right)^k \geq \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right\} - o(1)$$

and we also have

$$\exp \left\{ -\frac{\ell(\lambda)^2 d^2}{2^{d-2}} \right\} \geq 1 - o(1).$$

Putting these bounds into Lemma 4.1 we get

$$Z_\lambda(Q_d) \geq (2 - o(1))(1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right\}. \quad (28)$$

Noting that $\frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d = o(1)$ for λ satisfying (1), we get from (28) the claimed lower bounds on $Z_\lambda(Q_d)$ for λ satisfying either (1) or (2).

For λ satisfying either (3) or (4), $\frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d = \omega(1)$. For any $\varepsilon > 0$ set

$$f(\lambda) = \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d - \sqrt{(2 + \varepsilon) \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \log \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right)}$$

and

$$\ell(\lambda) = \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d + \sqrt{(2+\varepsilon) \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \log \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right)}.$$

An application of Corollary 3.4 yields

$$\sum_{k=f(\lambda)}^{\ell(\lambda)} \frac{1}{k!} \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right)^k \geq (1 - o(1)) \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right\}$$

and we also have

$$\exp \left\{ -\frac{\ell(\lambda)^2 d^2}{2^{d-2}} \right\} \geq \exp \left\{ -2d^2 \lambda^2 \frac{2^d}{(1+\lambda)^{2d}} \right\}.$$

Putting these bounds into Lemma 4.1 we get

$$Z_\lambda(Q_d) \geq (2 - o(1))(1+\lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d - 2d^2 \lambda^2 \frac{2^d}{(1+\lambda)^{2d}} \right\}. \quad (29)$$

Noting that

$$2d^2 \lambda^2 \frac{2^d}{(1+\lambda)^{2d}} = \begin{cases} o(1) & \text{for } \lambda \text{ satisfying (3)} \\ o\left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d\right) & \text{for } \lambda \text{ satisfying (4)} \end{cases} \quad (30)$$

we get from (29) the claimed lower bounds on $Z_\lambda(Q_d)$ for λ satisfying either (3) or (4).

Remark 4.4 *The lower bound*

$$Z_\lambda(Q_d) \geq (1+\lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d (1 - o(1)) \right\}$$

is valid (by the proof given above) for $\lambda > (1 + \Omega(1)) \log d/d$. For $\lambda \leq \log d/d$ it breaks down, as in this range the second inequality of (30) is no longer valid.

Proof of Lemma 4.3: For each $f \leq k \leq \ell$, we consider the contribution to the partition function from those I with $|I \cap \mathcal{E}| = k$, $e_1(k) \leq |I \cap \mathcal{O}| \leq e_2(k)$ and all 2-components of $I \cap \mathcal{E}$ having size 1, where $e_1(k) := \frac{\lambda}{1+\lambda} (2^{d-1} - dk) - \sqrt{\log d (2^{d-1} - dk)}$ and $e_2(k) := \frac{\lambda}{1+\lambda} (2^{d-1} - dk) + \sqrt{\log d (2^{d-1} - dk)}$. If we choose the elements of $I \cap \mathcal{E}$ sequentially then each new vertex we add removes from consideration at most $\binom{d}{2} + 1 \leq d^2$ vertices (those vertices which are at distance at most 2 from the chosen vertex). So the number of choices for $I \cap \mathcal{E}$ is at least

$$\frac{\prod_{j=0}^{k-1} (2^{d-1} - jd^2)}{k!} \geq \frac{1}{k!} \left(1 - \frac{\ell d^2}{2^{d-1}}\right)^\ell \geq \frac{1}{k!} \exp\left\{-\frac{\ell^2 d^2}{2^{d-2}}\right\}, \quad (31)$$

the second inequality using $1 - x \geq e^{-2x}$ for $0 < x < 1/2$; the application is valid since $\ell \leq \frac{2^{d-2}}{d^2}$.

Once $I \cap \mathcal{E}$ has been chosen, there are $2^{d-1} - dk$ vertices in \mathcal{O} from among which we choose between $e_1(k)$ and $e_2(k)$ to complete I . The sum of the weights of the valid extensions to I is, using Lemma 3.2,

$$\lambda^k \sum_{j=e_1(k)}^{e_2(k)} \lambda^j \binom{2^{d-1} - dk}{j} \geq \lambda^k (1 + \lambda)^{2^{d-1} - dk} \left(1 - \frac{2}{d^2}\right). \quad (32)$$

Combining (31) and (32) and noting that $e_1 \leq e_1(k)$ and $e_2(k) \leq e_2$ for all $f \leq k \leq \ell$, we see that the contribution to the partition function from those I with $f \leq |I \cap \mathcal{E}| \leq \ell$ and $e_1 \leq |I \cap \mathcal{O}| \leq e_2$ is at least

$$(1 + \lambda)^{2^{d-1}} \sum_{k=f}^{\ell} \frac{1}{k!} \left(\frac{\lambda}{2} \left(\frac{2}{1 + \lambda}\right)^d\right)^k \exp\left\{-\frac{\ell^2 d^2}{2^{d-2}}\right\} \left(1 - \frac{2}{d^2}\right).$$

We get at least the same contribution from those I with $f \leq |I \cap \mathcal{O}| \leq \ell$, $e_1 \leq |I \cap \mathcal{E}| \leq e_2$. Since $\ell < e_1$ there is no overlap between the two contributions, and all I under consideration satisfy $f \leq \min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq \ell$ and $e_1 \leq \max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq e_2$. This completes the proof of the lemma. \square

4.2 Proof of Theorem 1.1

The lower bounds on $Z_\lambda(Q_d)$ for λ satisfying (1), (2) and (3) come from considering only I satisfying

$$b_1(\lambda) \leq \max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} - \frac{\lambda 2^{d-1}}{1 + \lambda} \leq b_2(\lambda)$$

where

$$b_1(\lambda) = -d\ell(\lambda) - \sqrt{\log d (2^{d-1} - df(\lambda))}$$

and

$$b_2 = -df(\lambda) + \sqrt{\log d (2^{d-1} - df(\lambda))}$$

with $f(\lambda)$ and $\ell(\lambda)$ as introduced in the discussion after the statement of Lemma 4.3. For all such λ we have

$$b_1 \geq -2^{d/2} \sqrt{\log d} \quad \text{and} \quad b_2 \leq 2^{d/2} \sqrt{\log d},$$

the main point in both cases being that for λ satisfying (3), $d\lambda \left(\frac{2}{1+\lambda}\right)^d = o(2^{d/2})$. Since the lower bounds in this range are asymptotic to the upper bounds, that (5) occurs almost surely for this range of λ follows immediately, as does similarly the fact that (8) holds almost surely for λ satisfying (3).

That (6) holds almost surely for λ satisfying (1) follows immediately from Theorem 1.3. Indeed, the contribution to $Z_\lambda(Q_d)$ from those I with $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = 0$ is

$$2(1 + \lambda)^{2^{d-1}} - 1 \sim 2(1 + \lambda)^{2^{d-1}} \sim Z_\lambda(Q_d).$$

We have to work a little harder to show that (9) and (10) occur almost surely for λ satisfying (4). In this range, set

$$\mathcal{I}_\mathcal{E}(\lambda) = \left\{ I \in \mathcal{I}(Q_d) : \begin{array}{l} \text{cl}(I \cap \mathcal{E}) \leq m \\ \frac{1}{4 \log m} \frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d \leq k(I \cap \mathcal{E}) \leq em \frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d \\ \left| \max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} - \frac{\lambda 2^{d-1}}{1+\lambda} \right| \leq d \log d \left(\frac{2}{1+\lambda}\right)^d \end{array} \right\}.$$

where $\text{cl}(A)$ and $k(A)$ are the size of the largest 2-component of A and the number of 2-components of A , respectively, and define $\mathcal{I}_\mathcal{O}(\lambda)$ analogously. Note that $\mathcal{I}_\mathcal{E}(\lambda)$ and $\mathcal{I}_\mathcal{O}(\lambda)$ are disjoint and that $I \in \mathcal{I}_\mathcal{E}(\lambda)$ satisfies (9) and (10), so the following lemma completes the proof of Theorem 1.1.

Lemma 4.5 *For λ satisfying (4),*

$$Z_\lambda(Q_d) \sim \sum_{I \in \mathcal{I}_\mathcal{E}} \lambda^{|I|} + \sum_{I \in \mathcal{I}_\mathcal{O}} \lambda^{|I|}.$$

Before proving the lemma, it is worth pointing out that it provides significant structural information about the appearance of a typical independent set from Q_d chosen according to $\text{hc}(\lambda)$ for λ satisfying (4): it gives information about how many 2-components we expect the smaller of the two sides to have and how large each such 2-component is, as well as how large the larger of the two sides is. For λ satisfying (2) and (3) we can read similar information out of our proof of Theorem 1.3. Indeed, it is not hard to see that for λ satisfying (2), we obtain the conclusion of Lemma 4.5 if we replace “ $\text{cl}(I \cap \mathcal{E}) \leq m$ ” in the definition of $\mathcal{I}_{\mathcal{E}}$ with “ $\text{cl}(I \cap \mathcal{E}) \leq 1$ ” and we replace the condition on $k(I \cap \mathcal{E})$ with “ $k(I \cap \mathcal{E}) \leq \log d$ ” (say; any smaller $\omega(1)$ would also work here); and for λ satisfying (3) if we replace “ $\text{cl}(I \cap \mathcal{E}) \leq m$ ” with “ $\text{cl}(I \cap \mathcal{E}) \leq 1$ ” and we replace the condition on $k(I \cap \mathcal{E})$ with

$$\left| k(I \cap \mathcal{E}) - \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right| \leq \sqrt{(2+\varepsilon) \frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \log \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda} \right)^d \right)}.$$

(and in both cases we replace the condition on $|I \cap \mathcal{O}|$ with the upper and lower bounds from (5)).

Proof of Lemma 4.5: We begin by considering the contribution to $Z_{\lambda}(Q_d)$ from those I with $I \cap \mathcal{E}$ small and $\text{cl}(I \cap \mathcal{E}) > m$. With the sum below over such I , and recalling (25), we have

$$\begin{aligned} \sum \lambda^{|I|} &= (1+\lambda)^{2^{d-1}} \sum_{A \subseteq \mathcal{E} \text{ small, } \text{cl}(A) > m} \lambda^{|A|} (1+\lambda)^{-|N(A)|} \\ &\leq (1+\lambda)^{2^{d-1}} \sum_{A' \subseteq \mathcal{E} \text{ small, } 2\text{-linked, } |A'| > m} \lambda^{|A'|} (1+\lambda)^{-|N(A')|} \times \\ &\quad \sum_{A'' \subseteq \mathcal{E} \text{ small}} \lambda^{|A''|} (1+\lambda)^{-|N(A'')|} \\ &\leq Z_{\lambda}(Q_d) \sum_{A' \subseteq \mathcal{E} \text{ small, } 2\text{-linked, } |A'| > m} \lambda^{|A'|} (1+\lambda)^{-|N(A')|} \\ &= o(Z_{\lambda}(Q_d)), \end{aligned} \tag{33}$$

where in (33) we have used Corollary 3.8. We similarly have a negligible contribution to $Z_{\lambda}(Q_d)$ from those I with $I \cap \mathcal{O}$ small and $\text{cl}(I \cap \mathcal{O}) > m$.

Next we consider the contribution from those I with $I \cap \mathcal{E}$ small, $\text{cl}(I \cap \mathcal{E}) \leq$

m and $k(I \cap \mathcal{E}) \leq \frac{\lambda}{8 \log m} \left(\frac{2}{1+\lambda}\right)^d$. The contribution is at most

$$(1 + \lambda)^{2^{d-1}} \sum_{k \leq \frac{\lambda}{8 \log m} \left(\frac{2}{1+\lambda}\right)^d} \binom{2^{d-1}}{k} m^k \lambda^k (1 + \lambda)^{-dk} \quad (34)$$

$$\begin{aligned} &\leq (1 + \lambda)^{2^{d-1}} \sum_{k \leq \frac{\lambda}{8 \log m} \left(\frac{2}{1+\lambda}\right)^d} \frac{1}{k!} \left(\frac{\lambda m}{2} \left(\frac{2}{1+\lambda}\right)^d\right)^k \\ &\leq (1 + \lambda)^{2^{d-1}} \exp \left\{ (1 - \Omega(1)) \left(\frac{\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d\right) \right\} \quad (35) \\ &= o(Z_\lambda(Q_d)). \quad (36) \end{aligned}$$

The factor of $\binom{2^{d-1}}{k}$ in (34) counts the number of ways of choosing a fixed vertex in each of the k 2-components of $I \cap \mathcal{E}$. The factor of m^k counts the number of ways of assigning a size to each 2-component. For each choice of a fixed vertex and a size (ℓ_i , say) for each 2-component, the contribution to $Z_\lambda(Q_d)$ is at most

$$(1 + \lambda)^{2^{d-1}} \prod_{i=1}^k (ed^2)^{\ell_i-1} \lambda^{\ell_i} (1 + \lambda)^{-d\ell_i+2\ell_i(\ell_i-1)} \leq (1 + \lambda)^{2^{d-1}} \prod_{i=1}^k \lambda (1 + \lambda)^{-d}$$

(for large enough d , independent of λ). In (35) we use (14).

A similar calculation (using (15) in place of (14)) shows that the contribution from those I with $I \cap \mathcal{E}$ small, $\text{cl}(I \cap \mathcal{E}) \leq m$ and $k(I \cap \mathcal{E}) \geq \frac{em\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d$ is $o(Z_\lambda(Q_d))$, and by symmetry so too is the contribution from those I with $I \cap \mathcal{O}$ small, $\text{cl}(I \cap \mathcal{O}) \leq m$ and either $k(I \cap \mathcal{O}) \leq \frac{\lambda}{8 \log m}$ or $k \geq \frac{em\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d$.

We have shown that $\frac{1}{2}(1 - o(1))$ of $Z_\lambda(Q_d)$ comes from

$$\mathcal{I}'_{\mathcal{E}}(\lambda) = \left\{ I \in \mathcal{I}(Q_d) : \begin{array}{l} \text{cl}(I \cap \mathcal{E}) \leq m \\ \frac{\lambda}{8 \log m} \left(\frac{2}{1+\lambda}\right)^d \leq k(I \cap \mathcal{E}) \leq \frac{em\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d \end{array} \right\}$$

and another $\frac{1}{2}(1 - o(1))$ comes from the analogously defined $\mathcal{I}'_{\mathcal{O}}(\lambda)$. (We have dropped “ $I \cap \mathcal{E}$ small” since it is implied by the condition on $k(I \cap \mathcal{E})$).

It remains to show that the contribution to $\mathcal{I}'_{\mathcal{E}}(\lambda)$ from those I with $I \cap \mathcal{O}$ either too large or too small is negligible. For each $\frac{\lambda}{8 \log m} \left(\frac{2}{1+\lambda}\right)^d \leq$

$k \leq \frac{em\lambda}{2} \left(\frac{2}{1+\lambda}\right)^d$ and each choice of k 2-components A_1, \dots, A_k for $I \cap \mathcal{E}$, the contribution to $\sum_{I \in \mathcal{I}'_\mathcal{E}(\lambda)} \lambda^{|I|}$ is

$$\sum_{j=0}^{2^{d-1} - \sum_{i=1}^k |N(A_i)|} \lambda^j \binom{2^{d-1} - \sum_{i=1}^k |N(A_i)|}{j}.$$

By Lemma 3.2, all but a proportion at most $\frac{2}{d^2}$ of this sum comes from those j satisfying

$$\left| \frac{\lambda 2^{d-1}}{1+\lambda} - j \right| \leq \frac{\lambda}{1+\lambda} \sum_{i=1}^k |N(A_i)| + \sqrt{\log d \left(2^{d-1} - \sum_{i=1}^k |N(A_i)| \right)}.$$

For all k in the range under consideration, and all possible choices of the A_i 's, we have

$$\frac{\lambda}{1+\lambda} \sum_{i=1}^k |N(A_i)| + \sqrt{\log d \left(2^{d-1} - \sum_{i=1}^k |N(A_i)| \right)} \leq d \log d \left(\frac{2}{1+\lambda} \right)^d.$$

This completes the proof. \square

Remark 4.6 For small λ (say $\lambda < \sqrt{2} - 1 - \Omega(1)$) we get the better bound

$$\max\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq em^2 \lambda \left(\frac{2}{1+\lambda} \right)^d.$$

almost surely. An improvement in the range of λ for which Lemma 3.7 holds would lead to an improvement in the range for which this and (10) hold almost surely.

Finally, we turn to (7). Note that the right-hand side of (7) is

$$P(\text{Poisson}(\gamma_k) = c)$$

where $\text{Poisson}(\gamma_k)$ is a Poisson random variable with parameter $\gamma_k := \frac{1}{2}e^{-k/2}$.

For each fixed $c \in \mathbb{N}$ we get a lower bound on the contribution to the partition function from those I with $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = c$ by considering those which have c 2-components on \mathcal{E} , each of size 1, and have more than

$\log d$ (say) vertices on \mathcal{O} , and the same with \mathcal{E} and \mathcal{O} reversed. This gives a lower bound of

$$\frac{2}{c!} \prod_{i=0}^{c-1} (2^{d-1} - id^2) \lambda \left((1 + \lambda)^{2^{d-1}} - \sum_{i < \log d} \lambda^i \binom{2^{d-1} - cd}{\log d} \right)$$

which is at least

$$(2 - o(1))(1 + \lambda)^{2^{d-1}} \frac{1}{c!} \left(\frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d \right)^c. \quad (37)$$

Recalling the discussion just before the proof of Lemma 4.5, we know that all but a vanishing part of $Z_\lambda(Q_d)$ comes from I with the smaller of $I \cap \mathcal{E}$, $I \cap \mathcal{O}$ consisting of no more than $\log d$ 2-components of size 1. So we get an upper bound on the contribution to the partition function from those I with $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = c$ of

$$\binom{2^{d-1}}{c} \lambda^c (1 + \lambda)^{2^{d-1} - cd + o(1)} (Z_\lambda(Q_d)) \leq (2 + o(1))(1 + \lambda)^{2^{d-1}} \frac{1}{c!} \left(\frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d \right)^c \quad (38)$$

the inequality following from the fact that in this range of λ ,

$$Z_\lambda(Q_d) \sim 2(1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right) \right\} = O \left((1 + \lambda)^{2^{d-1}} \right).$$

Combining (37) and (38), and noting that for $\lambda \sim 1 + \frac{k}{d}$, $\frac{\lambda}{2} \left(\frac{2}{1 + \lambda} \right)^d \sim \gamma_k$, it follows that

$$P(\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} = c) \sim P(\text{Poisson}(\gamma_k) = c).$$

4.3 Proof of Theorem 1.5

Our approach is inspired by [5]. For typographic convenience we suppress the dependence on d in u_d , v_d and w_d . We begin with (12). Write

$$\mathcal{J} = \{J \in \mathcal{I}(Q_d) : w \in J\} \quad \text{and} \quad \mathcal{I}' = \{I \in \mathcal{J} : u \in I\}.$$

Further, write $\mathcal{I} = \{I \in \mathcal{I}' : I \cap \mathcal{E} \text{ small}\}$. We need to bound

$$\frac{w_\lambda(\mathcal{I}')}{w_\lambda(\mathcal{J})} \leq (1 + \lambda)^{-d(1 - o(1))}, \quad (39)$$

where $w_\lambda(*) = \sum_{I \in * } \lambda^{|I|}$. We will show

$$\frac{w_\lambda(\mathcal{I})}{w_\lambda(\mathcal{J})} \leq (1 + \lambda)^{-d(1-o(1))}. \quad (40)$$

The same argument will show

$$\frac{w_\lambda(\{I \in \mathcal{I} : I \cap \mathcal{O} \text{ small}\})}{w_\lambda(\{J \in \mathcal{I}(Q_d) : u \in \mathcal{J}\})} \leq (1 + \lambda)^{-d(1-o(1))}.$$

Combining this with (40) we get (39), noting that for any $I \in \mathcal{I}(Q_d)$, either $I \cap \mathcal{E}$ or $I \cap \mathcal{O}$ small, and that by symmetry $w_\lambda(\{J \in \mathcal{I}(Q_d) : u \in \mathcal{J}\}) = w_\lambda(\mathcal{J})$.

We will obtain (40) by producing, for each $I \in \mathcal{I}$, a set $\varphi(I) \subseteq \mathcal{J}$, as well as a map $\nu : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}$ supported on pairs (I, J) with $J \in \varphi(I)$ and satisfying

$$\sum_{J \in \varphi(I)} \nu(I, J) = 1 \quad (41)$$

for each $I \in \mathcal{I}$ and

$$\sum_{I \in \varphi^{-1}(J)} \lambda^{|I|-|J|} \nu(I, J) \leq (1 + \lambda)^{-d(1-o(1))} \quad (42)$$

for each $J \in \mathcal{J}$. It is not difficult to see that the existence of such a φ and ν satisfying (41) and (42) gives (40).

We produce φ as follows. Given $I \in \mathcal{I}$, write $W(I)$ for the 2-component of $I \cap \mathcal{E}$ containing u . Set

$$\mathcal{W}(a, g) = \{W \subseteq \mathcal{E} : |W| = a, |N(W)| = g, u \in W, W \text{ small and 2-linked}\}$$

and

$$\mathcal{I}(a, g) = \{I \in \mathcal{I} : W(I) \in \mathcal{W}(a, g)\}.$$

Set $I' = I \setminus W$. Note that $N(W(I)) \cap I = \emptyset$ and $N(W(I))$ is not adjacent to anything in I' (if it was, then $W(I)$ would not be the 2-component of u in $I \cap \mathcal{E}$). We may therefore add any subset of $N(W(I))$ to I' and still have an independent set. Set

$$\varphi(I) = \{I' \cup S : S \subseteq N(W(I))\}.$$

We have just observed that indeed $\varphi(I) \subseteq \mathcal{J}$.

For each $J \in \varphi(I)$ write $S(J)$ for $J \setminus I'$ and set

$$\nu(I, J) = \frac{\lambda^{S(J)}}{(1 + \lambda)^{|N(W(I))|}} \left(= \frac{\lambda^{|J| - |I| + |W(I)|}}{(1 + \lambda)^{|N(W(I))|}} \right).$$

Since $S(J)$ runs over all subsets of $N(W(I))$ it is clear that (41) holds. To see that (42) holds, observe that for fixed $J \in \mathcal{J}$ we have

$$\begin{aligned} \sum_{I \in \varphi^{-1}(J)} \lambda^{|I| - |J|} \nu(I, J) &= \sum_{I \in \varphi^{-1}(J)} \lambda^{|W(I)|} (1 + \lambda)^{-N(W(I))} \\ &\leq \sum_{a, g, W} \sum_{I \in \varphi^{-1}(J), I \in \mathcal{I}(a, g), W(I) = W} \lambda^a (1 + \lambda)^{-g} \\ &\leq \sum_{a, g} \sum_{W \in \mathcal{W}(a, g)} \lambda^a (1 + \lambda)^{-g}. \end{aligned} \quad (43)$$

The main point here is (43), which follows from the fact that for each $W \in \mathcal{W}(a, g)$ and $J \in \mathcal{J}$ there is at most one $I \in \mathcal{I}$ such that $I \in \mathcal{I}(a, g)$, $W(I) = W$ and $I \in \varphi^{-1}(J)$.

For each $g > d^4$ we have $\mathcal{W}(a, g) \subseteq \cup_{a' \leq g} \mathcal{G}(a', g)$, and so, using Lemma 3.7 for (44),

$$\begin{aligned} \sum_{a, g > d^4, W \in \mathcal{W}(a, g)} \lambda^a (1 + \lambda)^{-g} &\leq \sum_{g > d^4, a' \leq g} (1 + \lambda)^{-g} \sum_{A \in \mathcal{G}(a', g, u)} \lambda^{|A|} \\ &\leq 2^d \sum_{g > d^4, a' \leq g} (1 + \lambda)^{-\frac{c'(g-a') \log d}{d^{2/3}}}. \end{aligned} \quad (44)$$

By Lemma 3.6 we have $g - a' = \Omega(d^3)$ in the range $g > d^4$ and we have at most 2^d choices for each of a' and g and so

$$\sum_{a, g > d^4, W \in \mathcal{W}(a, g)} \lambda^a (1 + \lambda)^{-g} \leq (1 + \lambda)^{-d(1-o(1))}. \quad (45)$$

For $g \leq d^4$ we have $|\mathcal{W}(a, g)| \leq 2^{O(a \log d)} \leq 2^{O(g \log d/d)}$ and so

$$\begin{aligned} \sum_{a, g \leq d^4, W \in \mathcal{W}(a, g)} \lambda^a (1 + \lambda)^{-g} &\leq \sum_{a, g \leq d^4} (1 + \lambda)^{-g} 2^{O(g \log d/d)} (1 + \lambda)^{O(g/\log d)} \\ &\leq \sum_{g \geq d} (1 + \lambda)^{\left\{ O\left(\frac{\log d}{\log(1+\lambda)}\right) - g + O\left(\frac{g \log d}{d \log(1+\lambda)}\right) + \frac{g}{\log d} \right\}} \\ &\leq (1 + \lambda)^{-d(1-o(1))}. \end{aligned} \quad (46)$$

Combining (45) with (46) we obtain (40) and so (39) and (12).

We obtain (13) from (12) easily. Conditioned on $v \in I$, the probability that a particular odd neighbour of u is in I is, by (12), at most $(1+\lambda)^{-d(1-o(1))}$, and so the probability that none of the d neighbours of u are in I is at least $1 - d(1 + \lambda)^{-d(1-o(1))} = 1 - o(1)$. The probability that u is in I is at least the probability that it is in I conditioned on none of neighbours being in I times the probability that none of neighbours are in I , and so is at least $(1 - o(1))\lambda/(1 + \lambda)$.

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