

# Independent sets of a fixed size in the discrete hypercube

David Galvin\*

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## Abstract

We consider the problem of determining  $i_t(Q_d)$ , the number of independent sets of size  $t$  in the discrete hypercube  $Q_d = \{0, 1\}^d$ , for each  $0 \leq t \leq 2^{d-1}$ . We obtain asymptotically correct estimates as  $d \rightarrow \infty$  in the case when  $\liminf t(d)/2^{d-1}$  is greater than  $1 - 1/\sqrt{2}$ , and nearly matching upper and lower bounds otherwise. We use these estimates to obtain a partial unimodality result for the independent set (stable set) polynomial of  $Q_d$ , for large  $d$ .

## 1 Introduction and statement of results

For a (simple, undirected, loopless) graph  $G = (V, E)$  set

$$i_t(G) = \{I \in \mathcal{I}(G) : |I| = t\}$$

where  $\mathcal{I}(G)$  is the collection of independent sets of  $G$  (sets of vertices spanning no edges).

How large do we expect  $i_t(G)$  to be? For  $d$ -regular  $G$  an upper bound appears in [2]:

$$i_t(G) \leq \exp_2 \left\{ H \left( \frac{2t}{|V|} \right) \frac{|V|}{2} + \frac{|V|}{d} \right\} \quad (1)$$

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\*Department of Mathematics, University of Notre Dame, 255 Hurley Hall, Notre Dame IN 46556; dgalvin1@nd.edu.

where  $H(x) = -x \log_2 x - (1-x) \log_2(1-x)$  is the binary entropy function. In general this bound may be far from the truth, but for  $d$ -regular bipartite  $G$  it has the same leading term in the exponent as the trivial lower bound

$$i_t(G) \geq \binom{\frac{|V|}{2}}{t} \quad (2)$$

obtained by specifying a bipartition  $V = X \cup Y$  of  $G$  and considering only those independent sets which are subsets of  $X$  (note that by Stirling's formula there is a constant  $c > 0$  such that for all  $|V|$  and  $t$ ,

$$\log_2 \binom{\frac{|V|}{2}}{t} \geq H\left(\frac{2t}{|V|}\right) \frac{|V|}{2} - \frac{1}{2} \log_2 |V| - c.$$

The bound in (1) is obtained by considering

$$P(G, \lambda) = \sum_{t=0}^{\alpha(G)} i_t(G) \lambda^t$$

where  $\alpha(G)$  is the size of the largest independent set in  $G$ . This is the *independent set polynomial* or *stable set polynomial* of  $G$ , first introduced explicitly by Gutman and Harary [4]. For all  $t$  and  $\lambda > 0$ ,  $i_t(G) \lambda^t \leq P(G, \lambda)$  and so

$$i_t(G) \leq \min_{\lambda > 0} \left\{ \frac{P(G, \lambda)}{\lambda^t} \right\}.$$

From this we obtain (1) by taking  $\lambda = \frac{t}{2^{d-1}-t}$  in the following inequality, proved in [2]:

$$P(G, \lambda) \leq 2^{\frac{|V|}{d}} (1 + \lambda)^{\frac{|V|}{2}}. \quad (3)$$

The bound in (3) is valid for *all*  $d$ -regular graphs, and we expect that it, as well as both the upper and lower bounds on  $i_t(G)$  ((1) and (2)), can be significantly improved if further structural conditions are put on  $G$ . In [3] the programme of improving (3) is carried out in the case when  $G$  is the discrete hypercube  $Q_d$ . This is the graph on vertex set  $V = \{0, 1\}^d$  with two strings adjacent if they differ on exactly one coordinate. It is a  $d$ -regular bipartite graph with bipartition classes  $\mathcal{E}$  and  $\mathcal{O}$ , where  $\mathcal{E}$  is the set of vertices with an even number of 1's. Note that  $|\mathcal{E}| = |\mathcal{O}| = \alpha(Q_d) = 2^{d-1}$ . The following bound is obtained in [3].

**Theorem 1.1** *There is a  $c > 0$  and a function  $f(d) \rightarrow 0$  (as  $d \rightarrow \infty$ ) such that for  $\lambda > \frac{c \log d}{d^{1/3}}$ ,*

$$P(Q_d, \lambda) = 2(1 + \lambda)^{2^{d-1}} \exp \left\{ \frac{\lambda}{2} \left( \frac{2}{1 + \lambda} \right)^d (1 + f(d)) \right\}.$$

This generalizes work of Korshunov and Sapozhenko [6], who had shown  $P(Q_d, 1) = (2\sqrt{e} + o(1))2^{2^{d-1}}$ .

Using Theorem 1.1 (more correctly, using two of the intermediate inequalities that ultimately lead to the theorem), we can significantly improve the bounds on  $i_t(Q_d)$  given by (1) and (2). This is our main result.

**Theorem 1.2** *There is a constant  $c > 0$  such that if  $t = t(d)$  satisfies*

$$2^{d-1} \left( \frac{c \log d}{d^{1/3}} \right) \leq t \leq 2^{d-1} \left( 1 - \frac{1}{\sqrt{2}} + \frac{2 \log d}{d} \right) \quad (4)$$

for all sufficiently large  $d$  then

$$i_t(Q_d) = \binom{2^{d-1}}{t} \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} (1 + o(1)) \right\},$$

where  $o(1) \rightarrow 0$  as  $d \rightarrow \infty$ . If  $t$  satisfies

$$2^{d-1} \left( 1 - \frac{1}{\sqrt{2}} + \frac{2 \log d}{d} \right) \leq t \leq 2^{d-1} \quad (5)$$

for all sufficiently large  $d$  then

$$i_t(Q_d) \sim 2 \binom{2^{d-1}}{t} \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right\}$$

as  $d \rightarrow \infty$ .

A corollary of Theorem 1.2 is that the quantity  $i_t(Q_d)$  undergoes a transition around  $t = 2^{d-2}$ , in a window of width  $1/d$ . This is analogous to [3, Theorem 1.1].

**Corollary 1.3** *If  $t = t(d) = 2^{d-1} \left( \frac{1}{2} + \frac{f(d)}{d} \right)$  then*

$$\frac{i_t(Q_d)}{2^{\binom{2^{d-1}}{t}}} \rightarrow \begin{cases} \infty & \text{if } f(d) \rightarrow -\infty \\ \exp \{e^{-2k}/2\} & \text{if } f(d) \rightarrow k \\ 1 & \text{if } f(d) \rightarrow +\infty \end{cases}$$

as  $d \rightarrow \infty$ .

A sequence  $\{a_i\}_{i=0}^n$  is said to be *unimodal* (with *mode*  $k$ ) if

$$a_0 \leq a_1 \leq \dots \leq a_k \geq a_{k+1} \geq \dots \geq a_n,$$

and a polynomial  $\sum_{i=0}^n a_i x^n$  is said to be unimodal if the sequence of its coefficients is. Since (1) and (2) together suggest that for  $d$ -regular, bipartite  $G$  we have  $i_t(G) \approx 2^{\binom{|V|/2}{t}}$ , it is not unreasonable to suppose that for any such  $G$  the polynomial  $P(G, \lambda)$  is unimodal. Indeed, in [7] the stronger conjecture is made that  $P(G, \lambda)$  is unimodal if  $G$  is a König-Egerváry graph (a graph in which the size of the largest independent set plus the size of the largest matching equals the number of vertices in the graph; all bipartite graphs are König-Egerváry graphs). Very little progress has been made towards this conjecture; indeed, even the special case of the unimodality of  $P(T, \lambda)$  for  $T$  a tree (conjectured by Erdős et al. [1]) remains open.

A partial result from [7] is that for  $G$  a König-Egerváry graph,

$$i_{\lceil (2\alpha(G)-1)/3 \rceil}(G) \geq i_{\lceil (2\alpha(G)-1)/3 \rceil + 1}(G) \geq \dots \geq i_{\alpha(G)}(G).$$

By examining the error terms in the asymptotic estimates of  $i_t(Q_d)$  provided by Theorem 1.2, we are able to obtain a partial unimodality result for  $P(Q_d, \lambda)$  over a wider range of coefficients.

**Theorem 1.4** *For all sufficiently large  $d$  it holds that*

$$i_p(Q_d) < i_{p+1}(Q_d) < \dots < i_{2^{d-2} - 10^8 d^2}(Q_d)$$

where  $p = \lceil (1 - 1/\sqrt{2} + \frac{2 \log d}{d}) 2^{d-1} \rceil$ , and

$$i_{2^{d-2} + 10^8 d^2}(Q_d) > i_{2^{d-2} + 10^8 d^2 + 1}(Q_d) > \dots > i_{2^{d-1}}(Q_d).$$

The constant  $10^8$  is not the best that our argument implies; but since we cannot at present improve  $d^2$  to  $d^{2-\varepsilon}$  for any  $\varepsilon > 0$ , we see little value in optimizing it.

Section 2 presents two results from [3] that are needed for the proof of Theorem 1.2, while the proofs of Theorems 1.2 and 1.4 appear in Sections 3 and 4.

## 2 Preliminaries

We begin with some notation. For  $A \subseteq V (= \{0, 1\}^d)$  write  $N(A)$  for the set of vertices outside  $V$  that are neighbours of a vertex in  $A$  and set

$$[A] = \{v \in V : N(\{v\}) \subseteq N(A)\}.$$

Say that  $A \subseteq \mathcal{E}$  (or  $\mathcal{O}$ ) is *small* if  $|[A]| \leq 2^{d-2}$  and *2-linked* if  $A \cup N(A)$  induces a connected subgraph of  $Q_d$ . Any  $A$  can be decomposed into its maximal 2-linked subsets; we refer to these as the *2-components* of  $A$ , and write  $k(A)$  for the number of 2-components of  $A$  and  $\text{cl}(A)$  for the size of the largest 2-component of  $A$ .

There are two bounds from [3] (intermediate steps in the derivation of Theorem 1.1) that will be of use here. First, for  $\lambda > \frac{c \log d}{d^{1/3}}$  (for suitably large  $c$ ) we have

$$\sum_{A \subseteq \mathcal{E} \text{ small}} F_\lambda(|A|, |N(A)|) \leq \exp \left\{ \frac{\lambda}{2} \left( \frac{2}{1+\lambda} \right)^d + \frac{d^2 \lambda^2 (1+\lambda)^2 2^d}{(1+\lambda)^{2d}} \right\} \quad (6)$$

(this is [3, (27)]). Second, for fixed  $k \geq 1$  and  $\lambda > \frac{c \log d}{d^{1/3}}$ ,

$$\sum_{A \subseteq \mathcal{E} \text{ small, 2-linked, } |A| \geq k} F_\lambda(a, g) \leq e^{k-1} d^{2k-2} 2^d F_\lambda(k, kd - 2k(k-1)) \quad (7)$$

(this is [3, Corollary 3.8]).

## 3 Proof of Theorem 1.2

We assume throughout that  $t \geq \left(\frac{c \log d}{d^{1/3}}\right) 2^{d-1}$ , with  $c$  the same as the constant appearing in the range of validity of (6) (and (7)). We also, where necessary, assume that  $d$  is large enough to support our assertions. All asymptotic statements in what follows will be as  $d \rightarrow \infty$ .

For the lower bounds, we may also assume  $t \leq \frac{3}{4} 2^{d-1}$  (any constant greater than  $1/2$  would do in place of  $3/4$  here) since for  $t \geq \frac{3}{4} 2^{d-1}$ ,  $t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} = o(1)$  and so the bound  $i_t(Q_d) \geq (2 - o(1)) \binom{2^{d-1}}{t} \exp \left\{ t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} \right\}$  is triv-

ial. For the remaining range of  $t$ , the lower bounds will be based on

$$\begin{aligned}
i_t(Q_d) &\geq \sum_{A \subseteq \mathcal{E} \text{ or } A \subseteq \mathcal{O}, \text{cl}(A) \leq 1, |A| \leq f(t,d)} \binom{2^{d-1} - |N(A)|}{t - |A|} \\
&= 2 \sum_{A \subseteq \mathcal{E}, \text{cl}(A) \leq 1, |A| \leq f(t,d)} \binom{2^{d-1} - d|A|}{t - |A|} \tag{8}
\end{aligned}$$

where

$$f = f(t, d) = 100e \left( t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right).$$

The equality in (8) is valid since for all  $t$  under consideration,  $f < t/2$ , so there is no overlap between the independent sets of size  $t$  which intersect  $\mathcal{E}$  in no more than  $f$  vertices and those which intersect  $\mathcal{O}$  in no more than  $f$  vertices. (Note also that for  $A$  with  $\text{cl}(A) = 1$ ,  $|N(A)| = d|A|$ .)

For all  $a$  and  $g$  and  $0 \leq t \leq 2^{d-1}$ , we have

$$\binom{2^{d-1} - g}{t - a} = F_{\lambda(t)}(a, g) \binom{2^{d-1}}{t} E(a, g) \tag{9}$$

where  $\lambda(t) := \frac{t}{2^{d-1}-t}$  and

$$E(a, g) = \frac{\prod_{i=0}^{a-1} \left( 1 - \frac{i}{t} \right) \prod_{i=0}^{g-a-1} \left( 1 - \frac{i}{2^{d-1}-t} \right)}{\prod_{i=0}^{g-1} \left( 1 - \frac{i}{2^{d-1}} \right)}.$$

Note that

$$F_{\lambda(t)}(k, dk) = \left( \frac{t}{2^{d-1}} \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right)^k, \tag{10}$$

and that for fixed  $d$  and  $k$ , the function on the right-hand side in (10) is decreasing for all  $t \geq 2^{d-1}/d$  (a fact that we will use repeatedly in the calculations that follow, without further comment). For those  $A$  contributing to (8),

$$\begin{aligned}
E(|A|, d|A|) &\geq \prod_{i=0}^{f-1} \left( 1 - \frac{i}{t} \right) \prod_{i=0}^{(d-1)f-1} \left( 1 - \frac{i}{2^{d-1} - t} \right) \\
&\geq \exp \left\{ -\frac{f^2}{t} - \frac{d^2 f^2}{2^{d-1} - t} \right\}. \tag{11}
\end{aligned}$$

Here we use  $1 - x \geq e^{-2x}$  (valid for  $0 \leq x \leq 1/2$ ), together with  $f \leq t/2$  and  $df \leq (2^{d-1} - t)/2$ .

The number of ways of choosing  $A \subseteq \mathcal{E}$  with  $\text{cl}(A) \leq 1$  and  $|A| = k \leq f$  is at least

$$\frac{\prod_{i=0}^{k-1} (2^{d-1} - id^2)}{k!} \geq \frac{(2^{d-1})^k}{k!} \exp \left\{ -\frac{f^2 d^2}{2^{d-1}} \right\}, \quad (12)$$

since each choice of vertex in  $A$  eliminates from consideration at most  $d^2$  other vertices. By (9) each such  $A$  contributes

$$F_{\lambda(t)}(k, dk) \binom{2^{d-1}}{t} E(|A|, d|A|) \quad (13)$$

to the sum. Inserting (10), (11), (12) and (13) into (8) we get

$$i_t(Q_d) \geq 2 \binom{2^{d-1}}{t} E_0 \sum_{k \leq f} \frac{1}{k!} \left( t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right)^k \quad (14)$$

where

$$E_0 = \exp \left\{ -\frac{f^2}{t} - \frac{d^2 f^2}{2^{d-1} - t} - \frac{f^2 d^2}{2^{d-1}} \right\}$$

Noting that by our choice of  $f$

$$\sum_{k \leq f} \frac{1}{k!} \left( t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right)^k \geq \left( 1 - \frac{1}{100f} \right) \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right\}$$

(here we use  $k! \geq (k/e)^k$ ), (14) becomes

$$i_t(Q_d) \geq 2 \binom{2^{d-1}}{t} \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right\} E_1 \quad (15)$$

where  $E_1 = E_0 \left( 1 - \frac{1}{100f} \right)$ . For  $t$  satisfying (4)

$$E_1 \geq \exp \left\{ -o \left( t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right) \right\}$$

and for  $t$  satisfying (5)  $E_1 \geq 1 - o(1)$ , completing the lower bounds. We also note the following more precise bounds:

$$E_1 \geq \begin{cases} 1 - \frac{1}{d^3} & \text{if } t \text{ satisfies (5)} \\ 1 - \frac{10^8 d^2}{2^d} & \text{if } t \geq 2^{d-1} \left( \frac{1}{2} - \frac{1}{d} \right). \end{cases} \quad (16)$$

We now turn to the upper bounds. For any  $I \in \mathcal{I}(Q_d)$  we have  $[I \cap \mathcal{E}] \cap [I \cap \mathcal{O}] = \emptyset$ . Since  $Q_d$  has a perfect matching, it follows that at least one of  $[I \cap \mathcal{E}], [I \cap \mathcal{O}]$  is no larger than  $2^{d-2}$ , that is, that at least one of  $I \cap \mathcal{E}, I \cap \mathcal{O}$  is small. This together with  $\mathcal{E}$ - $\mathcal{O}$  symmetry leads to the bound

$$i_t(Q_d) \leq 2 \sum_{A \subseteq \mathcal{E} \text{ small}} \binom{2^{d-1} - |N(A)|}{t - |A|}. \quad (17)$$

We will examine the sum in (17) in three parts. Say that small  $A \subseteq \mathcal{E}$  is of *type I* if  $|A| \leq f$ , of *type II* if  $|A| > f$  and  $\text{cl}(A) \leq 5$ , and of *type III* if  $|A| > f$  and  $\text{cl}(A) \geq 6$ .

We first consider the contribution to the sum in (17) from  $A$  of type I. For these  $A$  we have

$$E(|A|, |N(A)|) \leq \exp \left\{ \frac{d^2 f^2}{2^{d-1}} \right\}$$

(this is similar to the derivation of (11), except that in this case we are lower bounding the numerator of  $E(|A|, |N(A)|)$ ). Taking  $\lambda = \frac{t}{2^{d-1}-t}$  in (6) and combining with (9) we find that the contribution to (17) from  $A$  of type I is at most

$$2 \binom{2^{d-1}}{t} \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} + \frac{d^2 t^2 2^d}{(2^{d-1} - t)^2} \left( 1 - \frac{t}{2^{d-1}} \right)^{2d-2} + \frac{d^2 f^2}{2^{d-1}} \right\}. \quad (18)$$

Next we consider the sum in (17) over  $A$  of type II. We have

$$\begin{aligned} \binom{2^{d-1} - g}{t - a} &= \lambda(t)^{a-t} \binom{2^{d-1} - g}{t - a} \lambda(t)^{t-a} \\ &\leq \lambda(t)^{a-t} (1 + \lambda(t))^{2^{d-1}-g} \\ &= (1 + \lambda(t))^{2^{d-1}} \lambda(t)^{-t} F_{\lambda(t)}(a, g) \\ &= 2^{H(\frac{t}{2^{d-1}})} 2^{d-1} F_{\lambda(t)}(a, g). \end{aligned}$$

By Stirling's formula, (more precisely, by the fact that for all  $n \geq 1$ ,

$$2n^n e^{-n} \sqrt{n} \leq n! \leq 3n^n e^{-n} \sqrt{n},$$

this is at most  $3\sqrt{2^d} \binom{2^{d-1}}{t} F_{\lambda(t)}(a, g)$ . It follows that the contribution to (17) from  $A$  of type II is at most

$$6\sqrt{2^d} \binom{2^{d-1}}{t} \sum_{A \subseteq \mathcal{E} \text{ small, } \text{cl}(A) \leq 5, k(A) \geq f/5} F_{\lambda(t)}(|A|, |N(A)|). \quad (19)$$

To bound this sum, we make a number of observations. First, for each  $k \geq f/5$  there are at most  $2^{k(d-1)}/k!$  ways of choosing a fixed vertex in each of the  $k$  2-components of  $A$ , and at most  $5^k$  ways of assigning a size to each 2-component. Next, for each  $\ell = 1, \dots, 5$ , the number of 2-linked subsets of  $\mathcal{E}$  of size  $\ell$  that include a fixed vertex is at most  $(\ell-1)!(d^2)^{\ell-1}$ , and also, each  $A \subseteq \mathcal{E}$  with  $|A| = \ell \leq 5$  satisfies  $|N(A)| \geq d\ell - 2\ell(\ell-1)$  (this is because pairs of vertices in  $Q_d$  that have common neighbours have exactly two such). Finally,  $F_\lambda(a, g)$  is decreasing in  $g$ , and for each  $\ell = 1, \dots, 5$ , (and sufficiently large  $d$ )

$$(\ell-1)!(d^2)^{\ell-1}F_{\lambda(t)}(\ell, d\ell - 2\ell(\ell-1)) \leq F_{\lambda(t)}(1, d).$$

All this together serves to bound the expression in (19) by

$$6\sqrt{2^d} \binom{2^{d-1}}{t} \sum_{k \geq f/5} \frac{1}{k!} \left( 5t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right)^k \leq \frac{2}{4^{f/5}} \binom{2^{d-1}}{t}. \quad (20)$$

This follows from the choice of  $f$  and the bound  $k! \geq (k/e)^k$ .

Finally we consider the sum in (17) over  $A$  of type III. Beginning with the same steps as in the case of  $A$  of type II, this is at most

$$6\sqrt{2^d} \binom{2^{d-1}}{t} \sum_{A \subseteq \mathcal{E} \text{ small, } \text{cl}(A) \geq 6} F_{\lambda(t)}(|A|, |N(A)|). \quad (21)$$

We now use a multiplicative property of  $F$ : for any  $\lambda$ , if  $A'$  is a 2-component of  $A$  then

$$F_\lambda(|A|, |N(A)|) = F_\lambda(|A'|, |N(A')|)F_\lambda(|A \setminus A'|, |N(A \setminus A')|)$$

and so the sum in (21) is at most

$$\begin{aligned} & \sum_{A \subseteq \mathcal{E} \text{ small, 2-linked, } |A| \geq 6} F_{\lambda(t)}(|A|, |N(A)|) \sum_{A \subseteq \mathcal{E} \text{ small}} F_{\lambda(t)}(|A|, |N(A)|) \\ & \leq e^5 d^{10} 2^d F_{\lambda(t)}(6, 6d - 60) \times \\ & \quad \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} + \frac{d^2 t^2 2^d}{(2^{d-1} - t)^2} \left( 1 - \frac{t}{2^{d-1}} \right)^{2d-2} \right\}, \end{aligned} \quad (22)$$

with (22) using both (6) and (7). Combining (18), (20) and (22) we find that

$$i_t(Q_d) \leq 2 \binom{2^{d-1}}{t} \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right\} E_2 \quad (23)$$

where

$$E_2 = \exp \left\{ \frac{d^2 t^2 2^d}{(2^{d-1} - t)^2} \left( 1 - \frac{t}{2^{d-1}} \right)^{2d-2} + \frac{d^2 f^2}{2^{d-1}} \right\} + \frac{2}{4f/5} + 3e^5 d^{10} 2^d \sqrt{2^d} F_{\lambda(t)}(6, 6d - 60) \exp \left\{ \frac{d^2 t^2 2^d}{(2^{d-1} - t)^2} \left( 1 - \frac{t}{2^{d-1}} \right)^{2d-2} \right\}.$$

For  $t$  satisfying (4)

$$E_2 \leq \exp \left\{ o \left( t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right) \right\}$$

and for  $t$  satisfying (5)  $E_2 \leq 1 + o(1)$ , completing the upper bounds. We also note the following more precise bounds:

$$E_2 \leq \begin{cases} 1 + \frac{1}{d^3} & \text{if } t \text{ satisfies (5)} \\ 1 + \frac{10^8 d^2}{2^d} & \text{if } t \geq 2^{d-1} \left( \frac{1}{2} - \frac{1}{d} \right). \end{cases} \quad (24)$$

## 4 Proof of Theorem 1.4

Fix  $2^{d-2} + 10^8 d^2 \leq t < 2^{d-1}$ . We will show that for large  $d$  (independent of  $t$ )

$$i_t(Q_d) > i_{t+1}(Q_d).$$

By (15), (16), (23) and (24) it is enough to show that

$$\frac{2 \left( 1 - \frac{10^8 d^2}{2^d} \right) \binom{2^{d-1}}{t} \exp \left\{ t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1} \right\}}{2 \left( 1 + \frac{10^8 d^2}{2^d} \right) \binom{2^{d-1}}{t+1} \exp \left\{ (t+1) \left( 1 - \frac{t+1}{2^{d-1}} \right)^{d-1} \right\}} > 1.$$

Since  $t \left( 1 - \frac{t}{2^{d-1}} \right)^{d-1}$  is decreasing in  $t$ , this inequality is implied by

$$\frac{\left( 1 - \frac{10^8 d^2}{2^d} \right) (t+1)}{\left( 1 + \frac{10^8 d^2}{2^d} \right) (2^{d-1} - t)} > 1$$

which holds for all  $t \geq 2^{d-2} + 10^8 d^2$  (for large enough  $d$ , independent of  $t$ ).

Next, fix  $2^{d-1} \left(1 - \frac{1}{\sqrt{2}} + \frac{2 \log d}{d}\right) \leq t < 2^{d-1} \left(\frac{1}{2} - \frac{1}{d}\right)$ . Now we wish to show that for large  $d$  (independent of  $t$ )

$$\frac{2 \left(1 + \frac{1}{d^3}\right) \binom{2^{d-1}}{t} \exp \left\{ t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} \right\}}{2 \left(1 - \frac{1}{d^3}\right) \binom{2^{d-1}}{t+1} \exp \left\{ (t+1) \left(1 - \frac{t+1}{2^{d-1}}\right)^{d-1} \right\}} < 1.$$

Using

$$\begin{aligned} & \exp \left\{ t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} - (t+1) \left(1 - \frac{t+1}{2^{d-1}}\right)^{d-1} \right\} \\ & \leq \exp \left\{ t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} \left(\frac{2(d-1)}{2^{d-1} - t}\right) \right\} \leq 1 + \frac{1}{2^{d/3}} \end{aligned}$$

this is implied by

$$\frac{\left(1 + \frac{2}{d^3}\right) (t+1)}{\left(1 - \frac{1}{d^3}\right) (2^{d-1} - t)} < 1$$

which holds for all  $t$  under consideration (for large  $d$ , independent of  $t$ ).

Finally, in the range  $2^{d-1} \left(\frac{1}{2} - \frac{1}{d}\right) \leq t < 2^{d-2} - 10^8 d^2$  we wish to show

$$\frac{2 \left(1 + \frac{10^8 d^2}{2^d}\right) \binom{2^{d-1}}{t} \exp \left\{ t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} \right\}}{2 \left(1 - \frac{10^8 d^4}{2^d}\right) \binom{2^{d-1}}{t+1} \exp \left\{ (t+1) \left(1 - \frac{t+1}{2^{d-1}}\right)^{d-1} \right\}} < 1.$$

Using

$$\exp \left\{ t \left(1 - \frac{t}{2^{d-1}}\right)^{d-1} - (t+1) \left(1 - \frac{t+1}{2^{d-1}}\right)^{d-1} \right\} \leq 1 + \frac{d^2}{2^d}$$

this is implied by

$$\frac{\left(1 + \frac{(10^8+1)d^4}{2^d}\right) (t+1)}{\left(1 - \frac{10^8 d^4}{2^d}\right) (2^{d-1} - t)} < 1$$

which holds for all  $t$  under consideration (for large  $d$ , independent of  $t$ ).

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