Math 60850, Spring 2016, Homework 3

1) Rosenthal 4.3.2

a) Given $X \leq Y$

- If ω is s.t. $X(\omega), Y(\omega) \leq 0$,
  then $X^+(\omega) = Y^+(\omega) = 0$
- If ω is s.t. $X(\omega) \leq 0$, $Y(\omega) > 0$,
  then $0 = X^+(\omega) \leq Y^+(\omega) = Y(\omega)$
- If ω is s.t. $X(\omega), Y(\omega) \geq 0$,
  then $X^+(\omega) = X(\omega) \leq Y(\omega) = Y^+(\omega)$
- Since $X \leq Y$, not possible for
  $X(\omega) > 0$, $Y(\omega) \leq 0$.

Conclusion: $X^+ \leq Y^+$ always

Similar case-by-case argument shows
$$X^- \geq Y^-$$

b) Using non-negativity of X^+, X^-, Y^+, Y^-, and order-preserving property of non-negative r.v.s, have $E(X) = E(X^+) - E(X^-) \leq E(Y^+) - E(Y^-) = E(Y)$.

2) Rosenthal 4.3.3

a) $\left.\begin{array}{l} X = X^+ - X^- \\ Y = Y^+ - Y^- \end{array}\right\}$ $Z = X + Y = X^+ - X^- + Y^+ - Y^-$

$$\begin{array}{c} Z \\ \parallel \\ Z^+ - Z^- \end{array}$$

b) From a), $Z^+ + X^- + Y^- = Z^- + X^+ + Y^+$

Using linearity for non-negative rvs,

$$E(Z^+) + E(X^-) + E(Y^-) = E(Z^-) + E(X^+) + E(Y^+),$$

So

$$\begin{aligned} E(Z) &= E(Z^+) - E(Z^-) \\ &= E(X^+) + E(Y^+) - E(X^-) - E(Y^-) \\ &= E(X) + E(Y) \end{aligned}$$

c) Follows from b) by induction.

3) Rosenthal 4.3.4

a) We know that if X, Y ind, and f, g are Borel measurable functions, then

$f(X), g(Y)$ are independent

Apply this with $f = x\mathbb{1}_{\{x \geq 0\}}$

$g = y\mathbb{1}_{\{y \geq 0\}}$

to conclude that $\underbrace{X\mathbb{1}_{\{X \geq 0\}}}_{= X^+}, \underbrace{Y\mathbb{1}_{\{Y \geq 0\}}}_{= Y^+}$ are ind

Similar for the three other pairs.

b) $Z = Z^+ - Z^- = (X^+ - X^-)(Y^+ - Y^-) \quad (= XY)$

$= X^+Y^+ - X^+Y^- - X^-Y^+ + X^-Y^-$

c) By linearity, and then independence

$$E(Z) = E(X^+Y^+) - E(X^+Y^-) - E(X^-Y^+) + E(X^-Y^-)$$

$$= E(X^+)E(Y^+) - E(X^+)E(Y^-)$$

$$- E(X^-)E(Y^+) + E(X^-)E(Y^-)$$

$$= (E(X^+) - E(X^-))(E(Y^+) - E(Y^-)) = E(X)E(Y).$$

4) Cauchy-Schwarz-Bunyakovsky:

$$E(X^2), E(Y^2) < \infty$$

$$E(|XY|) \leq \sqrt{E(X^2)E(Y^2)}$$

Apply this to $X_1 = X - E(X)$

$Y_1 = Y - E(Y)$

Have $E(X_1^2) = Var(X) < \infty$

$E(Y_1^2) = Var(Y) < \infty$

So by CSB,

$$E(|(X-E(X))(Y-E(Y))|) \leq \sqrt{Var(X)Var(Y)}$$

But now by triangle inequality,

$$|E((X-E(X))(Y-E(Y)))| \leq \sqrt{Var(X)Var(Y)},$$

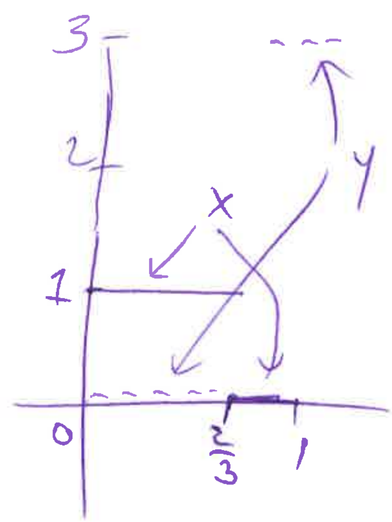
ie $|Cov(X,Y)| \leq \sqrt{Var(X)Var(Y)}$,

so $|Corr(X,Y)| \leq 1$, as required.

5) Rosenthal 4.5.3

Many possibilities, eg

$$X(\omega) = \begin{cases} 1 \text{ if } \omega \in [0, \tfrac{2}{3}) \\ 0 \text{ if } \omega \in [\tfrac{2}{3}, 1] \end{cases}$$



$$Y(\omega) = \begin{cases} 0 \text{ if } \omega \in [0, \tfrac{2}{3}) \\ 3 \text{ if } \omega \in [\tfrac{2}{3}, 1] \end{cases}$$

$$\left.\begin{aligned} E(X) &= \tfrac{2}{3} \\ E(Y) &= 1 \end{aligned}\right\} \quad E(X) < E(Y)$$

$$\{\omega \mid X(\omega) > Y(\omega)\} = [0, \tfrac{2}{3}) \Big\} \; P(X>Y) = \tfrac{2}{3} > \tfrac{1}{2}$$

6) Rosenthal 5.5.1

$$P(X \geq 3) = P(2^X \geq 2^3)$$
$$= P(Y \geq 8) \quad [Y = 2^X]$$
$$= P(Y \geq 2E(Y))$$
$$\leq \frac{1}{2} \quad [\text{Markov, since } Y \geq 0]$$

7) Rosenthal 5.5.9

$$P(X-m \geq \alpha) = P(X-m+y \geq \alpha+y) \quad (y \geq 0)$$

$$\leq P(X-m+y \geq \alpha+y) + P(X-m+y \leq -\alpha-y)$$

$$= P(|X-m+y| \geq \alpha+y)$$

$$= P((X-m+y)^2 \geq (\alpha+y)^2)$$

$$\leq \frac{E((X-m+y)^2)}{(\alpha+y)^2} \quad \text{(Markov)}$$

$$= \frac{E((X-m)^2 + 2y(X-m) + y^2)}{(\alpha+y)^2}$$

$$= \frac{Var(X) + 2y\,\overset{0}{\cancel{E(X-m)}} + y^2}{(\alpha+y)^2}$$

$$= \frac{Var(X) + y^2}{(\alpha+y)^2}$$

Plug in $y = \frac{v}{\alpha}$ to get $\frac{Var(X)+y^2}{(\alpha+y)^2} = \frac{v + \frac{v^2}{\alpha^2}}{(\alpha + \frac{v}{\alpha})^2}$

$$= \frac{v}{v+\alpha^2} \quad \left[\begin{array}{l}\text{after some} \\ \text{algebra}\end{array}\right]$$

So $P(X-m \geq \alpha) \leq \frac{v}{v+\alpha^2}$