MATH 60850, Homework 1

SPRING 2016

1) [Rosenthal 2.7.1]

a) $\mathcal{F}_1$ is a σ-algebra

b) $\mathcal{F}_2$ is a σ-algebra

$\mathcal{F}_1, \mathcal{F}_2$ both have same structure:

A a set, $A_1 \cup A_2 \cup \ldots \cup A_\ell = A$ a partition

$\mathcal{F}$ consists of all sets of the form

$$\bigcup_{i \in I} A_i ,$$

where I runs over all subsets of $\{1, \ldots, \ell\}$

[$\mathcal{F}_1$: $\{1,2,3,4\} = \{1,2\} \cup \{3,4\}$

$\mathcal{F}_2$: $\{1,2,3,4\} = \{1,2\} \cup \{3\} \cup \{4\}$]

Call such a σ-algebra an <u>atomic</u> σ-algebra generated by the partition; it will come up again later in the homework.

c) $\mathcal{F}_3$ is not a σ-algebra; $\{1,2\} \cup \{1,3\} \notin \mathcal{F}_3$.

2) [Rosenthal 1.3.5]

a) The only place we used that probability of an interval = length of the interval was in saying $P([0,1]) = 1$

b) The function $R : 2^{[0,1]} \to [0,1]$ given by $R(A) = 0 \ \forall A$ is countably additive and shift invariant

[It is the unique such function : following the proof of Prop 1.2.6, we find that if R is countably additive and shift invariant, then $R([0,1]) = \sum_{h=1}^{\infty} R(H)$.
For R to have range in $\mathbb{R}^+$, we need $R(H) = 0$, whence $R([0,1]) = 0$
Monotonicity (which follows from non-negativity and ctble additivity) forces $R(A) = 0 \ \forall A \subseteq [0,1]$.
Shift invariance then forces $R(\{0\}) = 0$, and then additivity finally gives $R(A) = 0 \ \forall A \subseteq [0,1)$]

3) [Rosenthal 2.7.7]

$\mathcal{F} = \{A \subseteq [0,1] :$ either A ctble or A^c ctble$\}$

a) • $\phi \in \mathcal{F}$ ✓

and b) • If $A \in \mathcal{F}$ and A ctble, then A^c has property that $(A^c)^c$ is ctble, so $A^c \in \mathcal{F}$.

Similar if $A \in \mathcal{F}$ and A^c ctble.

So $\mathcal{F}$ closed under complementation ✓

• If $A_1, A_2, \dots \in \mathcal{F}$, and all are ctble, then $\bigcup_{i=1}^{\infty} A_i$ ctble [a countable union of countable sets is countable]

So $\bigcup A_i \in \mathcal{F}$.

If even one of the A_i, A_1 say, has A_1^c countable, then

$(\bigcup A_i)^c = \bigcap A_i^c \subseteq A_1^c$, so is ctble,

and in this case also $\bigcup A_i \in \mathcal{F}$

So $\mathcal{F}$ is closed under ctble union ✓

Conclusion: $\mathcal{F}$ is a σ-algebra, and so also $\mathcal{F}$ is an algebra.

c) Suppose $A, A_1, A_2, \ldots \in \mathcal{F}$ with

and d) $A = \cup A_i$, A_i's disjoint

Case i): all A_i ctble, so A ctble.

Here $P(A) = 0$ and $\Sigma P(A_i) = 0$,

So $P(A) = \Sigma P(A_i)$

Case ii): A_1 has A_1^c ctble,

each of $A_2, A_3, \ldots$ ctble

In this case $A_1 \cup A_2 \ldots = A$ has

property that A^c ctble [as in parts a), b)],

So $P(A) = 1$ and $\sum_{i=1}^{\infty} P(A_i) = 1 + 0 + 0 + \ldots = 1$,

and $P(A) = \sum_{i=1}^{\infty} P(A_i)$

Case iii) At least two of the A_i's, say

A_1, A_2, have countable complements.

This case <u>cannot</u> occur: Since A_1, A_2

disjoint, $A_2 \subseteq A_1^c$, but A_2 is uncountable

and A_1^c is countable.

Conclusion: P is countable additive, and so

is also finitely additive.

4) [Rosenthal 2.3.16]

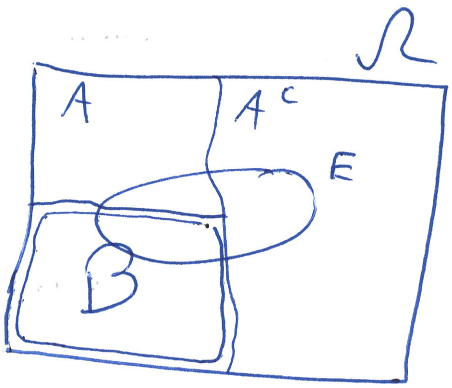
Given $(\Omega, \mathcal{M}, P^*)$ coming from pair $(\mathcal{J}, P)$ via extension theorem.

Let $A \in \mathcal{M}$ satisfy $P^*(A) = 0$, and let $B \subseteq A$ be given.

Want to conclude $B \in \mathcal{M}$.

Fix $E \subseteq \Omega$, and consider

$$P^*(E \cap B) + P^*(E \cap B^c)$$



$E \cap B \subseteq A$ so by monotonicity of P^*, $\overset{①}{\boxed{P^*(E \cap B) = 0}}$

Also, $E \cap A^c \subseteq E \cap B^c \subseteq E$, so

$$P^*(E \cap A^c) \leq P^*(E \cap B^c) \leq P^*(E) \quad (*)$$

But now, since $A \in \mathcal{M}$ have $P^*(E) = P^*(E \cap A) + P^*(E \cap A^c)$,

and since $E \cap A \subseteq A$ have $P^*(E \cap A) = 0$, so

$P^*(E \cap A^c) = P^*(E)$

Using this (*) becomes

$$\rho^*(E) \leq \rho^*(E \cap B^c) \leq \rho^*(E),$$

So $\boxed{\rho^*(E \cap B^c) = \rho^*(E)}$ 2

Hence $\rho^*(E \cap B) + \rho^*(E \cap B^c) = \rho^*(E)$,

Combining 1 and 2 and $B \in \mathcal{M}$.

5) [Rosenthal 2.7.4]

a) $(\mathcal{F}_i)_{i=1}^{\infty}$ a sequence of algebras,

$\mathcal{F}_i \subseteq \mathcal{F}_{i+1} \ \forall i$

Claim: $\cup \mathcal{F}_i$ is an algebra.

Proof: • $\phi \in \mathcal{F}_1$ so $\phi \in \cup \mathcal{F}_i$

• Suppose $A \in \cup \mathcal{F}_i$. Then $A \in \mathcal{F}_n$ for some n, so $A^c \in \mathcal{F}_n$ [$\mathcal{F}_n$ is an algebra]

So $A^c \in \cup \mathcal{F}_i$, $\cup \mathcal{F}_i$ closed under complementation.

• Suppose $A_1, \dots, A_l \in \cup \mathcal{F}_i$.

Because the $\mathcal{F}_i$'s are nested, we know that $\exists n$ s.t. $A_1, \dots, A_l \in \mathcal{F}_n$

[take eg $n = \max\{n_1, \dots, n_l\}$, where n_i is chosen s.t. $A_i \in \mathcal{F}_{n_i}$]

Since $\mathcal{F}_n$ is an algebra, $\bigcup_{i=1}^{l} A_i \in \mathcal{F}_n$,

so $\bigcup_{i=1}^{l} A_i \in \cup \mathcal{F}_i$, and $\cup \mathcal{F}_i$ is closed under finite union

Conclusion: $\cup \mathcal{F}_i$ is an algebra.

b) Let $\mathcal{F}_n$ be the atomic σ-algebra generated by the partition $\mathbb{N} = \{1\} \cup \{2\} \cup \ldots \cup \{n\} \cup \{m > n\}$.

It is evident that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$

(since the partition defining $\mathcal{F}_{n+1}$ is a refinement of the partition defining $\mathcal{F}_n$)

<u>But</u>: We have $\{1\}, \{3\}, \{5\}, \{7\}, \ldots$ all in $\bigcup_{n=1}^{\infty} \mathcal{F}_n$, whereas we do <u>not</u> have

$\{1, 3, 5, 7, \ldots\}$ $\rightarrow$ anything in $\bigcup \mathcal{F}_n$ is in $\mathcal{F}_\ell$ for some ℓ, and so is either finite (which $\{1, 3, 5, \ldots\}$ is not), or includes all integers bigger than ℓ (which $\{1, 3, 5, \ldots\}$ does not).

Conclusion: $\bigcup \mathcal{F}_n$ is <u>not</u> a σ-algebra.