MATH 60850, Spring 2016. Midterm

1) a) Given $A \in \mathcal{F}$ w/ $P(A) \le \frac{1}{2}$, immediately have $Q(A) = P(A)$.

Given $A \in \mathcal{F}$ w/ $P(A) > \frac{1}{2}$, have $P(A^c) = 1 - P(A) < \frac{1}{2}$,

so $P(A^c) = Q(A^c)$

ie $1 - P(A) = 1 - Q(A)$, $P(A) = Q(A)$.

b) Many possibilities. Perhaps simplest:

$\Omega = \{1, 2\}$

$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$P(\emptyset) = 0$	$Q(\emptyset) = 0$
$P(\{1\}) = P(\{2\}) = \frac{1}{2}$	$Q(\{1\}) = \frac{1}{3}$
$P(\{1,2\}) = 1$	$Q(\{2\}) = \frac{2}{3}$
	$Q(\{1,2\}) = 1$

2) a) Ω

b) $[0,1]$ (or any set which contains $[0,1]$); $\mathcal{F}$

c) Ω; $\{\omega \in \Omega \mid X(\omega) < 5\}$

d) $\mathbb{R}$; Ω [also acceptable: Range (X); Ω]

e) Measurable (or Borel, or Borel Measurable);
$\mathbb{R}$ (or Range (X)); $\mathcal{F}$

3) a) Suppose $\omega \in \limsup (A_n \cap B_n)$.

Then there are infinitely many indices n such that $\omega \in A_n \cap B_n$.

Hence there are ∞ly many indices such that $\omega \in A_n$ (the same set of indices for which $\omega \in A_n \cap B_n$), and there are ∞ly many indices such that $\omega \in B_n$.

So $\omega \in \limsup A_n$

and

$\omega \in \limsup B_n$

i.e., $\omega \in \limsup A_n \cap \limsup B_n$.

So $(\limsup A_n) \cap (\limsup B_n) \supseteq \limsup (A_n \cap B_n)$.

b) For an example where inclusion holds with equality, let $(\Omega, \mathcal{F})$ be arbitrary, let $A \in \mathcal{F}$, and set $A_i = A = B_j$ $\forall i, j$.

For an example where inclusion is strict, let $(\Omega, \mathcal{F}, P)$ be countable infinite fair coin tossing.

Let A_n be the event of heads on n^{th} toss
let B_n " " " " tails " " ".

Then $A_n \cap B_n = \emptyset$ for all n,

So $\limsup (A_n \cap B_n) = \emptyset$.

But any $\omega \in \Omega$ that has both ∞ly many 1's and ∞ly many 0's (eg, $(1,0,1,0,\ldots)$)
is in $(\limsup A_n) \cap (\limsup B_n)$,

So $(\limsup A_n) \cap (\limsup B_n)$ has strictly more elements than $\limsup (A_n \cap B_n)$.

4) a) Set $A = \{X > 0\}$.

For each $n \in \mathbb{N}$, set $A_n = \{X \geq \frac{1}{n}\}$

Note that $A_1 \subseteq A_2 \subseteq \dots$ and that

$\bigcup_{n=1}^{\infty} A_n = A$, so by continuity,

$\lim_{n \to \infty} P(A_n) = P(A) = \varepsilon > 0$ (hypothesis)

Since $\lim_{n \to \infty} P(A_n) = \varepsilon > 0$, there exists

$N \in \mathbb{N}$ s.t for all $n \geq N$, $P(A_n) \geq \frac{\varepsilon}{2}$.

In particular, $P(A_N) \geq \frac{\varepsilon}{2}$.

So $P(X \geq \frac{1}{N}) > 0$, take $\delta = \frac{1}{N}$.

b) let $A_n = \{\omega \mid X_n(\omega) > n\}$

$$\subseteq \{\omega \mid |X_n(\omega)| > n\}$$

$$P(A_n) \leq P(|X_n| > n)$$

$$\leq \frac{1}{n^2} \quad \text{(by Chebeshev's inequality)}$$

So $\sum_{n=1}^{\infty} P(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$.

So by Borel-Cantelli (1),

$$P(\limsup A_n) = P(X_n > n \text{ i.o.}) = 0$$

5) a) $X_n \to 0$ in probability

$\iff$

$\forall \varepsilon > 0,\ P(|X_n| > \varepsilon) \to 0$

$\iff$

$\forall \varepsilon \in (0, \frac{1}{2}),\ P(|X_n| > \varepsilon) \to 0$

($\Rightarrow$ trivial

for $\Leftarrow$, if $P(|X_n| > \frac{1}{4}) \to 0$,

then for all $\varepsilon \geq \frac{1}{2}$, since

$\{|X_n| > \varepsilon\} \subseteq \{|X_n| > \frac{1}{4}\}$,

have $P(|X_n| > \varepsilon) \to 0$)

$\iff$

$\forall \varepsilon \in (0, \frac{1}{2}),\ p_n \to 0$

$\iff$

$p_n \to 0$

(didn't need independence here)

b) First suppose that $\sum_{n=1}^{\infty} p_n < \infty$.

Fix $\varepsilon > 0$.

$$P(|X_n| \geq \varepsilon) = \left\{ \begin{array}{ll} 0 & \text{if } \varepsilon > 1 \\ p_n & \text{if } \varepsilon \leq 1 \end{array} \right\} \leq p_n$$

So $\sum_{n=1}^{\infty} P(|X_n| \geq \varepsilon) \leq \sum_{n=1}^{\infty} p_n < \infty$.

By Borel-Cantelli (1),

$$P(|X_n| \geq \varepsilon \text{ i.o.}) = 0$$

By lemma 5.2.1 we conclude that

$$X_n \to 0 \text{ almost surely.}$$

(again, we haven't used independence)

Conversely, suppose $\sum_{n=1}^{\infty} p_n = \infty$

Set $A_n = \{\omega \,/\, X_n(\omega) = 1\}$

By independence of the X_n, the A_n's are independent.

$P(A_n) = p_n$ so $\sum_{n=1}^{\infty} P(A_n) = \infty$.

Hence by Borel-Cantelli (2),

$$P(\limsup A_n) = 1$$

For $\omega \in \limsup A_n$, $X_n \not\to 0$

Hence $P(X_n \not\to 0) = 1$,

$X_n \not\to 0$ almost surely.

6) a) $\sigma(H_2, \ldots,)$ is the smallest σ-algebra that includes $H_2, \ldots$; formally it is the intersection of all σ-algebras that include $H_2, \ldots$. If we can find a single σ-algebra that includes $H_2, \ldots$, but does not include H_1, then we are done, since then necessarily $\sigma(H_2, \ldots)$ (being a subset of what we have just constructed) does not contain H_1.

So: Recall $\Omega = \{0,1\}^{\mathbb{N}}$ $(\mathbb{N} = \{1, 2, \ldots\})$

Write this as $\{0,1\} \times \{0,1\}^{\mathbb{N}^{\geq 2}}$

$(\mathbb{N}^{\geq 2} = \{2, 3, \ldots\})$

For each $A \subseteq \{0,1\}^{\mathbb{N}^{\geq 2}}$, set

$$A' = \{(0,w) \mid w \in A\} \cup \{(1,w) \mid w \in A\}$$

(i.e, take all strings in A, representing the results of coins tosses 2 on, and get strings in Ω by appending both 0 at the begining, and 1).

Then it is clear:

1) $\mathcal{F} = \{A' \mid A \subseteq \{0,1\}^{\mathbb{N}^{\geq 2}}\}$

is a σ-algebra on Ω

2) $H_2, H_3, \ldots \in \mathcal{F}$

3) $H_1 \notin \mathcal{F}$

So $H_1 \notin \sigma(H_2, H_3, \ldots)$

b) The claim is trivially false. For example, take $A_i = \phi$ for all i.