Discrete Mathematics, Spring 2009: Homeworks

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1. Lots of people were at the Mall in DC on Tuesday, and there was much hugging. Prove that there were two people who were each involved in the same number of hugs. (Assume that all hugs involved exactly two people.)

   Solution: n people, each hugged some number between 0 and $n-1$; **but**, it's not possible for someone to have hugged 0 people and someone else to have hugged $n-1$; so there are $n-1$ possible hug-numbers (either 0 through $n-2$ or 1 through $n-1$) distributed among n people; by pigeon-hole principle, at least two people must get the same number.

2. The *multinomial coefficient* $\binom{n}{a_1,a_2,\ldots,a_k}$ (with $\sum_i a_i = n$ and each $a_i \geq 0$) is defined by

$$\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1!a_2!\ldots a_k!}.$$

   (a) Give a combinatorial interpretation (beginning "it's the number of lists of length n from an alphabet of size $k\ldots$").

   Solution: It's the number of lists of length n from an alphabet of size k in which the ith letter occurs a_i times.

   (b) Give a combinatorial argument for the following identity, called the *Multinomial Theorem*: for all $x_1, \ldots, x_k \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{a_1+a_2+\ldots+a_k=n,\ a_i\geq 0} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}.$$

   (Notice that this reduces to the binomial theorem when $k = 2$.)

   Solution: When expand out the left hand side as

$$(x_1 + x_2 + \ldots + x_k)(x_1 + x_2 + \ldots + x_k)\ldots(x_1 + x_2 + \ldots + x_k) = F_1F_2\ldots F_k,$$

   the number of times that the term $x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}$ occurs is equal to the number of lists of length n from alphabet $\{x_1, \ldots, x_k\}$ in which the x_i occurs a_i times (the positions in which x_i occurs corresponding to the factors in which x_i is the chosen term).

(c) Give a combinatorial argument for the following identity, a generalization of Pascal's identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$:

$$\binom{n}{a_1, a_2, \ldots, a_k} = \binom{n-1}{a_1 - 1, a_2, \ldots, a_k} + \binom{n-1}{a_1, a_2 - 1, \ldots, a_k} + \ldots + \binom{n-1}{a_1, a_2, \ldots, a_k - 1}.$$

Solution: The jth term on the right-hand side the number of lists of length n from an alphabet of size k in which the ith letter occurs a_i times, *and* with the jth letter occurring in the first position. Since *some* letter must occur in the first position, when we sum we get the number of lists of length n from an alphabet of size k in which the ith letter occurs a_i times.

3. The *polynomial principle* is the assertion that if two polynomials (in variable x) agree at infinitely many values (for example, at all positive integers), then they are equal for all values of x.

(a) Use this to give a new combinatorial proof of the binomial theorem. (Begin by pulling out a factor of x^n from both sides, so that both sides become polynomials of a single variable, $z := y/x$. Then check that both sides count the same thing for each $z \in \mathbb{N}$.)

Solution: The case $x = 0$ of the binomial theorem is trivial, so it's ok to divide both sides by x^n. After the suggested transformation, we're left with

$$(1 + z)^n = \sum_{i=0}^{n} \binom{n}{i} z^{n-i}.$$

For each positive integer z, the left-hand side counts the number of n-words on alphabet $[1 + z]$. The ith term on the right-hand side counts the number of n-words on alphabet $[1 + z]$ in which the letter 1 occurs i times ($\binom{n}{i}$ choices for the location of the 1's and z^{n-i} choices for filling letters into the remaining $n - i$ slots using only the z letters $\{2, \ldots, 1 + z\}$). Summing over i, we get all n-words on alphabet $[1 + z]$. So the identity holds for all positive integers; by the polynomial principle, the identity holds for all z.

(b) The polynomial principle also holds for polynomials of many variables; do West 1.1.30.

Solution: Induction on k; base case ($k = 1$) is assumed. Viewed as a polynomial in x_1, $p(x_1, \ldots, x_k)$ is a polynomial of degree at most d_1 whose coefficients are polynomials in $x_2, \ldots, x_k$ with the maximum degree of x_i being at most d_i; say $p(x) = p_1 x^{d_1} + p_2 x^{d_1 - 1} + \ldots + p_{d_1}$. For each fixed $(x_2, \ldots, x_k) \in \prod_{i=2}^{k} S_i$, $p(x)$ reduces to a polynomial with real coefficients that vanishes at $d_1 + 1$ places, so is identically zero. It follows that each of $p_1, \ldots, p_{d_1}$ vanishes on $\prod_{i=2}^{k} S_i$, and so by induction are identically zero. Thus the original polynomial is identically zero.

If a polynomial of several variables $(x_1, \ldots, x_k)$ vanishes at all positive integers, then in particular it vanishes on some set of the form $\prod_{i=1}^{k} S_i$ where S_i is a set of size exceeding the maximum degree of $x)i$ appearing in the polynomial, so it is identically zero.

4. (a) Draw the tree whose Prüfer Code is $(6, 1, 2, 1, 2, 1, 6)$.

   Solution: ...

   (b) A *star* is a tree in which one vertex is joined to all the others. What are the possible Prüfer Codes of a star on n vertices?

   Solution: The star in which the label of the central vertex is i ($i = 1, \dots, n$) has Prüfer Code $(i, i, i, \dots, i)$. (Notice that there are n stars, and we've listed n Prüfer Codes.)

   (c) A *path* is a tree in which two vertices have degree 1 and all the rest have degree 2. What are the possible Prüfer Codes of a path on n vertices?

   Solution: There are $n!/2$ such trees. By the degree characterization of Prüfer Codes, each path has a list of $n-2$ distinct symbols as its code (a list of all the degree 2 vertices, in some order). There are $\binom{n}{n-2}(n-2)! = n!/2$ such lists, so the set of Prüfer Codes of paths must be exactly the set of simple $(n-2)$-words on alphabet $[n]$.

5. West 1.1.18

   Solution: There are four types of lists: those beginning and ending with a run of 1's; those beginning and ending with a run of 0's; those beginning with a run of 1's and ending with a run of 0's; and those beginning with a run of 0's and ending with a run of 1's. We'll count the first of these. There is a bijection between [lists of m 1's and n 0's containing exactly k runs of 1's and beginning and ending with a run of 1's (and therefore also containing $k-1$ runs of 0's)] and [pairs of compositions, the first of m into k parts, the second of n into $k-1$ parts]. It follows that the number of lists of the first type is

$$\binom{m-1}{k-1}\binom{n-1}{k-2}.$$

   Using similar observations for the other three parts, we find that the total number of lists is

$$\binom{m-1}{k-1}\binom{n-1}{k-2} + 2\binom{m-1}{k-1}\binom{n-1}{k-1} + \binom{m-1}{k-1}\binom{n-1}{k}$$

   or

$$\binom{m-1}{k-1}\left(\binom{n-1}{k-2} + 2\binom{n-1}{k-1} + \binom{n-1}{k}\right).$$

   Above is how I thought of the problem. Many of you gave a simpler (also correct) expression:

$$\binom{m-1}{k-1}\binom{n+1}{k}.$$

6. West 1.1.31

   Solution: Fix positive integers x and y. The left-hand sides counts the number of simple n-words on alphabet $[x+y]$. The ith term on the right-hand side counts the number of such words in which i of the letters come from $[x]$ and the remaining letters from $\{x+1, \dots, x+$

$y\}$. Since each word must include i letters from $[x]$ for some $i = 1, \ldots, n$, it follows that the right-hand side also counts simple n-words on alphabet $[x+y]$, so the two sides are equal. Since they are equal for all positive integers, they are identical (by the polynomial principle).

7. West 1.1.36

   Solution: It's
   $$\sum_{i=0}^{k} \text{compositions of } i \text{ into } n \text{ parts} = \sum_{i=0}^{k} \binom{i-1}{n-1}.$$
   By a binomial coefficient summation identity, this simplifies to $\binom{k}{n}$ (think of counting subsets of $[k]$ of size n by first fixing the largest element). To see this simpler expression directly, notice that by adding a (positive) dummy variable x_{n+1} to the equation, we can biject the set of solutions to the compositions of $k+1$ into $n+1$ parts: there are $\binom{n}{k}$ such.

   To see the answer even more directly, note that the following is a bijection from solutions to $\sum_{i=1}^{n} x_i = k$ in positive integers and subset of $[k]$ of size n:
   $$(x_1, \ldots, x_n) \to \{x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_k\}.$$
   The map is clearly injective (look at the first place where two compositions differ) and it is also surjective, since the set $\{a_1, \ldots, a_n\}$ (with $a_1 < \ldots < a_n$) is the image of $(a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1})$.

8. West 1.2.13a)

   Solution: I'll do induction on n. The identity is clearly true for $n = 0$ and all values of k. For $n > 0$
   $$\begin{aligned}
   \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\
   &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
   &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} + \frac{1}{n-k}\right) \\
   &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{n}{k(n-k)}\right) \\
   &= \frac{n!}{k!(n-k)!}.
   \end{aligned}$$

9. West 1.2.13c)

   Solution: Life is a lot easier here if we avail of the fact that $\binom{n}{k}$ is 0 if $k > n$ or $k < 0$. This allows us to write the binomial theorem as
   $$(x+y)^n = \sum_{k} \binom{n}{k} x^k y^{n-k}$$

where the sum is unrestricted (except to integral k).

Again, we'll do induction on n, with the case $n = 0$ trivial. For $n > 0$

$$\begin{aligned}
\sum_k \binom{n}{k} x^k y^{n-k} &= \sum_k \binom{n-1}{k} x^k y^{n-k} + \sum_k \binom{n-1}{k-1} x^k y^{n-k} \\
&= y \sum_k \binom{n-1}{k} x^k y^{(n-1)-k} + x \sum_k \binom{n-1}{k-1} x^{k-1} y^{(n-1)-(k-1)} \\
&= y(x+y)^{n-1} + x(x+y)^{n-1} \\
&= (x+y)^n.
\end{aligned}$$

10. West 1.2.25

    Solution: It's clear by inspection that $a_1 = 1$, $a_2 = 1+6$, $a_3 = 1+6+12$, $a_4 = 1+6+12+18$ and in general $a_n = 1 + 6\sum_{i=1}^{n-1} i = 1 + 6\binom{n-1}{2}$, so that $\sum_{k=1}^n a_k = n + 6\sum_{k=1}^n \binom{k-1}{2}$. Use a summation identity, this is $n + 6\binom{n+1}{3} = n^3$. As for a bijection ...

11. West 1.2.28

    Solution: Here's a general identity, that generalizes the binomial theorem:

    $$\prod_{i=1}^n (x_i + y_i) = \sum_{S \subseteq [n]} \prod_{i \notin S} x_i \prod_{i \in S} y_i.$$

    (To prove this, just think of the choices we make as we run though the n factors on the left while expanding out the product).

    With all $x_i = 1$, and $y_i = 1/i$, the identity becomes

    $$\sum_{S \subseteq [n]} \prod_{i \in S} \frac{1}{i} = \prod_{i=1}^n \left(1 + \frac{1}{i}\right) = n + 1.$$

    With all $x_i = 1$, and $y_i = -1/i$, the identity becomes

    $$\sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \in S} \frac{1}{i} = \prod_{i=1}^n \left(1 - \frac{1}{i}\right) = 0.$$

    NB: By convention, the product over an empty set is 1 (rather than 0).

12. West 1.2.38

    Solution: We count the pairs (X, Y) in two ways:

    (a) Choose X of size k from M, then Y of size m from $X \cup N$; total $\binom{m}{k}\binom{n+k}{m}$. Summing over k we get that the left-hand side counts the number of pairs.

(b) Choose $Y \cap N$ of size k from N, then $M \setminus Y$ from M (necessarily of size k, since we must have $|Y| = m$), then choose $X \setminus Y$ arbitrarily from $M \setminus Y$; total $\binom{n}{k}\binom{m}{k}2^k$. Summing over k we get that the left-hand side counts the number of pairs.

13. We defined the *extended Prüfer code* of a labeled tree T on vertex set $\{1, \ldots, n\}$ as follows: set $T_0 = T$; for $i = 1, \ldots n-1$, let b_i be the label of the lowest labeled leaf of T_{i-1}, and a_i the label of its neighbour; let T_i be the tree obtained from T_{i-1} by deleting the lowest labeled leaf and its edge; then set $a = (a_1, \ldots, a_{n-1})$ and $b = (b_1, \ldots, b_{n-1})$. We then defined the *Prüfer code* of T to be the string $a' = (a_1, \ldots, a_{n-2})$. We observed

    - T is reconstructible from its extended Prüfer code
    - $a_{n-1} = n$ always
    - for $i = 0, \ldots, n-2$, the Prüfer code of T_i is just $(a_{i+1}, \ldots, a_{n-2})$
    - the number of occurrences of label i in the Prüfer code is one less than the degree of i in the tree
    - the extended Prüfer code (and therefore the tree T) is reconstructible from the Prüfer code via the following procedure:
        - $a = (a_1, \ldots, a_{n-2}, n)$
        - for $i = 1, \ldots, n-1$, b_i is the least label not appearing in $\{b_1, \ldots, b_{i-1}\} \cup \{a_i, \ldots, a_{n-2}\}$
    - Each labeled tree on $\{1, \ldots, n\}$ has a distinct Prüfer code consisting of a string of length $n-2$ from the alphabet $\{1, \ldots, n\}$.

    Complete the proof of Cayley's Theorem by proving that if $(a_1, \ldots, a_{n-2})$ is a string of length $n-2$ from the alphabet $\{1, \ldots, n\}$, and we set $a_{n-1} = n$ and (for $i = 1, \ldots, n-1$) b_i to be the least label not appearing in $\{b_1, \ldots, b_{i-1}\} \cup \{a_i, \ldots, a_{n-2}\}$, then the set of pairs $\{\{a_1, b_1\}, \{a_1, b_1\}, \ldots, \{a_{n-1}, b_{n-1}\}\}$ form the edges of a tree on $\{1, \ldots, n\}$ whose Prüfer code is $(a_1, \ldots, a_{n-2})$.

14. Here's a series of short exercises that leads to a celebrated conclusion. n is any integer here, and p is always a prime number.

    (a) Give a combinatorial proof that $\binom{2n}{n} \geq \binom{2n}{k}$ for all n, k. **Hint**: exhibit an *injection* from the subsets of size k (of a set of size $2n$) to the subsets of size n.

    (b) Conclude from part a) that $\binom{2n}{n} \geq \frac{4^n}{2n+1}$. **Hint**: consider the expression $(1+1)^{2n}$.

    (c) Defining $o_p(n)$ to be the highest power of the prime p that divides n, verify the (simple) identities $o_p(ab) = o_p(a) + o_p(b)$ and $o_p(a/b) = o_p(a) - o_p(b)$. (Assume for the second that $b|a$, that is, that a/b is an integer.)

    (d) Using part c), show that if $2n/3 < p \leq n$ then $o_p(\binom{2n}{n}) = 0$.

    (e) Show that $o_p(n!) = \sum_{i \geq 1} [n/p^i]$.

(f) Show that if $p|\binom{2n}{n}$, then $p^{o_p(\binom{2n}{n})} \leq 2n$. **Hint**: let $r(p)$ be such that $p^{r(p)} \leq 2n < p^{r(p)+1}$. Argue that $o_p(\binom{2n}{n}) \leq r(p)$ using parts $c)$ and $e)$.

(g) Show that $\binom{2m+1}{m} \leq 2^{2m}$ for all m. **Hint**: use the binomial theorem and the identity $\binom{n}{k} = \binom{n}{n-k}$.

(h) Show that $\prod_{m+2 \leq p \leq 2m+1} p \leq \binom{2m+1}{m}$ where the product is over primes. **Hint**: this is easy and doesn't involve any calculation.

(i) Show that $\prod_{p \leq n} p \leq 4^n$ for all n (where the product is over primes). **Hint**: Use induction on n; you'll probably need to deal with the cases n even and n odd separately. For n odd ($n = 2m + 1$), parts g) and h) might be helpful.

Now write $\binom{2n}{n} = ABC$ where A consists only of prime factors that are less than $\sqrt{2n}$, B consists only of prime factors that are between $\sqrt{2n}$ and $2n/3$, and C consists only of prime factors that are greater than n and at most $2n$. (By part d), there are no prime factors of $\binom{2n}{n}$ between $2n/3$ and n).

(j) Show that $A \leq 2n^{\sqrt{2n}}$. **Hint**: At most how many prime factors can be involved in A? By part f), at most how much can each prime factor contribute to A?

(k) Show that $B \leq 4^{2n/3}$. **Hint**: use part f) to figure out what is $o_p(\binom{2n}{n})$ for the p contributing to B, and then use part i).

(l) Show that $C \geq \frac{4^{n/3}}{(2n+1)2n^{\sqrt{2n}}}$. **Hint**: combine parts b), j) and k).

(m) Show that for all sufficiently large n, $C > 1$.

Bertrand's Postulate is the statement that for all $n \geq 1$, there is a prime number p satisfying $n < p \leq 2n$. It was verified by Bertrand in 1845 for $n < 3000000$, and proved for all n by Tchebychev in 1850 (using complex analysis).

(n) Conclude from part m) that Bertrand's postulate is true for all *large* n.

Maple tells me that the right hand side of the inequality in part l) is greater than 1 for all $n \geq 468$, and so the argument outlined in this exercise verifies Bertrand's postulate for all $n \geq 468$. This beautiful elementary argument appeared in Paul Erdős' first paper (1932).

(o) 467 cases of Bertrand's postulate remain to be verified ($n = 1, 2, \ldots, 467$). Find a quick argument that deals with all of these cases.

15. Complete the characterization of solutions to a linear, finite order recurrence by proving the following: if $\alpha_1, \ldots, \alpha_r$ are distinct complex numbers (all non-zero) and $d_1, \ldots, d_r$ are non-negative integers, then the sequences

$$< \alpha_1^n >, < n_{(1)}\alpha_1^n >, \ldots, < n_{(d_1-1)}\alpha_1^n >, < \alpha_2^n >, < n_{(1)}\alpha_2^n >, \ldots,$$
$$< n_{(d_2-1)}\alpha_1^n >, \ldots, < \alpha_r^n >, < n_{(1)}\alpha_r^n >, \ldots, < n_{(d_r-1)}\alpha_r^n >$$

are linearly independent. (Recall that $< \alpha_1^n >$ is shorthand for the sequence $(1, \alpha_1, \alpha_1^2, \alpha_1^3, \ldots)$, etc.)

Solution: Consider a possible linear relation:

$$\lambda_{10} < \alpha_1^n > + \lambda_{11} < n_{(1)}\alpha_1^n > + \ldots + \lambda_{r(d_r-1)} < n_{(d_r-1)}\alpha_r^n > = < 0 > .$$

We claim by induction on $\sum_{i=1}^r d_i$ that all the λ's must be 0. Dotting both sides with $< x^n >$ we get

$$\frac{\lambda_{10}}{1-\alpha_1 x} - \frac{\lambda_{11}\alpha_1^2 x^2}{(1-\alpha_1 x)^2} + \ldots + \frac{\lambda_{jk}(-1)^k k! \alpha_j^{2k} x^k}{(1-\alpha_j x)^k} + \ldots + \frac{\lambda_{r(d_r-1)}(-1)^{d_r-1}(d_r-1)!\alpha_r^{2(d_r-1)} x^{d_r-1}}{(1-\alpha_r x)^{d_r-1}} = 0$$

the equality valid for all x inside in a suitable small circle around the origin (the circle $|x| < \frac{1}{|\alpha_1|}$, if we assume wlog that α_1 is the largest of the α_i's in magnitude). Putting the whole expression under a common denominator $\prod_{i=1}^r (1-\alpha_i x)^{d_i-1}$ we get a polynomial in the numerator that includes the term:

$$\lambda_{1(d_1-1)}(-1)^{d_1-1}(d_1-1)!\alpha_1^{2(d_1-1)} x^{d_1-1} \prod_{i=2}^r (1-\alpha_i x)^{d_i-1}.$$

All other terms include $(1-\alpha_1 x)$ as a factor. If we evaluate at $x = 1/\alpha_1$ (valid? take limit as $x \to 1/\alpha_1$) we get $\lambda_{1(d_1-1)} = 0$; the result follows by induction.

16. The Delannoy numbers clearly satisfy the recurrence relation

$$d_{n,m} = d_{n-1,m} + d_{n,m-1} + d_{n-1,m-1},\ n, m \geq 1$$

with initial conditions $d_{n,0} = 1 = d_{0,m}$ for all $n, m \geq 0$.

Let $a_{n,m}$ denote the number of points in the Hamming ball of radius m in $\mathbb{Z}^n$. By showing that the $a_{n,m}$ satisfy the same recurrence relation (with the same boundary conditions) as the Delannoy numbers, show that $d_{n,m} = a_{m,n}$ for all $n, m \geq 0$.

Solution: This is West Proposition 2.1.10 and Remark 2.1.11 (page 70).

17. In class we established a recurrence relation for the Catalan numbers: $C_0 = 1$, $C_n = \sum_{k=1}^n C_{n-k}C_{k-1}$ for $n \geq 1$. Let T_n be the number of triangulations of a convex $(n+2)$-gon. Show that $T_n = C_n$ by arguing that the T_n's satisfy the same recurrence relation as the C_n's. **Hint**: Focus attention on one side of the $(n+2)$-gon. Which triangle of the triangulation does this side belong to?

Solution: This is West Remark 2.1.18 (page 68).

18. Wilf Chapter 1, question 11.

Solution: Wilf page 199.

19. Wilf Chapter 1, question 21 a).

Solution: The nth derivative of e^{e^x} is

$$e^{e^x}\sum_k S(n,k)e^{kx} = e^{e^x}\sum_{k=0}^{n} S(n,k)e^{kx}$$

where $S(n,k)$ is the Stirling number of the second kind. We can prove this by induction. It's certainly true for $n=0$ (remember that $S(0,0)=1$). For $n>0$, the nth derivative is the derivative of the $(n-1)$st derivative (which is by induction $e^{e^x}\sum_k S(n-1,k)e^{kx}$). So the nth derivative is

$$\begin{aligned} & e^{e^x}\sum_k kS(n-1,k)e^{kx} + e^{e^x}\sum_k S(n-1,k)e^{(k+1)x} \text{ (product rule)} \\ = \; & e^{e^x}\sum_k \left(kS(n-1,k)+S(n-1,k-1)\right)e^{kx} \text{ (reindexing second sum)} \\ = \; & e^{e^x}\sum_{k=0}^{n} S(n,k)e^{kx} \text{ (basic recurrence satisfied by } S\text{'s).} \end{aligned}$$

20. West 2.1.19

    Solution: Recall that $\hat{F}_k$ counts the number of compositions of n into parts that are either size 1 or 2; that is, the number of solutions to $\sum_i x_i = n$ with each $x_i = 1$ or 2. Such a composition of $n+m$ falls into one of two categories: those in which there is some j with $\sum_{i\leq j} x_i = n$ and $\sum_{i>j} x_i = m$, and those in which there is no such j. There are $\hat{F}_n\hat{F}_m$ compositions of the first kind (since any such can be decomposed into an initial segment which is a decomposition of n and a terminal segment which is a decomposition of m). There are $\hat{F}_{n-1}\hat{F}_{m-1}$ compositions of the second kind (since any such can be decomposed into an initial segment which is a decomposition of $n-1$, followed by a part of size 2, followed by a terminal segment which is a decomposition of $m-1$). So $\hat{F}_{n+m} = \hat{F}_n\hat{F}_m + \hat{F}_{n-1}\hat{F}_{m-1}$.

    A special case of this is

    $$\hat{F}_{kn-1} = \hat{F}_{n-1}\hat{F}_{(k-1)n} + \hat{F}_{n-2}\hat{F}_{(k-1)n-1}, (\star)$$

    valid for $k \geq 1$. We can now prove that $\hat{F}_{n-1}|\hat{F}_{kn-1}$ for all $k\geq 1$, by induction on k. It's trivial for $k=1$, and for $k>1$, since (by induction) $\hat{F}_{n-1}$ divides both terms of the rhs of $(\star)$, it divides the lhs.

21. West 2.1.43 b)

    Solution: Clearly $a_1 = 1$. For $n>1$, let $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}$ be the points. What can x_{11} be paired up with? Something of the form x_{k2}, for some $k=1,\ldots,n$; because if x_{11} is paired with x_{k1} then the remaining points are partitioned into two sets each of odd size (those that fall to one side of the line joining x_{11} and x_{k1} and those on the other), so no completion is possible. If x_{11} is joined to x_{k2}, how many completions are possible? a_{k-1} ways to pair up the points x_{12} through x_{k1} and, independently, a_{n-k} ways to pair up the remaining points. So $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$, the Catalan recurrence (with the Catalan initial condition $a_0 = 1$).

22. West 2.2.13

Solution: We begin with $a_0 = 0$. For $n > 0$: where does the first 1 appear? In one of positions $k = 1, \dots, n$. In the $k-1$ spaces before the first 1, we should fill in an arbitrary word on alphabet $\{3,4\}$; there are 2^{k-1} such words. In the remaining $n-k$ positions, we should fill in an arbitrary word on alphabet $\{1,2,3,4\}$; there are 4^{n-k} such words. So

$$a_n = \sum_{k=1}^{n} 2^{k-1}4^{n-k} = \frac{4^n}{2}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k = \frac{4^n}{2}\left(1-\left(\frac{1}{2}\right)^n\right) = 2^{2n-1} - 2^{n-1} = (2^n-1)2^{n-1}.$$

A direct argument: Let A be the set of indices at which a 1 or a 2 appear. For each choice for A, there are $2^{|A|-1}$ choices for the 1's and 2's (the first index in A must be a 1, but the rest are free), and $2^{n-|A|}$ choices for for the 3's and 4's, so 2^{n-1} choices in all. And, since A is not empty, there are $2^n - 1$ choices for A.

23. West 2.2.34

Solution: Let $D(x,y) = \sum_{m,\, n\geq 0} d_{m,n}x^m y^n$ be the generating function. Using the recurrence and the initial conditions we get

$$D(x,y) = 1 + xD(x,y) + yD(x,y) + xyD(x,y)$$

and so

$$\begin{aligned}
D(x,y) &= \frac{1}{1-x-y-xy} \\
&= \frac{1}{(1-x)(1-y)\left(\frac{1-x-y-xy}{(1-x)(1-y)}\right)} \\
&= \frac{1}{(1-x)(1-y)\left(1-\frac{2xy}{(1-x)(1-y)}\right)} \\
&= \frac{1}{(1-x)(1-y)}\sum_{k\geq 0} 2^k \frac{x^k}{(1-x)^k}\frac{y^k}{(1-y)^k} \\
&= \sum_{k\geq 0} 2^k \frac{x^k}{(1-x)^{k+1}}\frac{y^k}{(1-y)^{k+1}} \\
&= \sum_{k\geq 0} 2^k \sum_{s\geq 0}\binom{s}{k}x^s \sum_{t\geq 0}\binom{t}{k}y^t
\end{aligned}$$

What's the coefficient of $x^m y^n$ in this? From each $k \geq 0$ we get a contribution of $2^k\binom{m}{k}\binom{n}{k}$, and so

$$d_{m,n} = \sum_{k\geq 0} 2^k \binom{m}{k}\binom{n}{k}.$$

24. West 3.2.7a)

    Solution: Set $a_n = \sum_i 2i\binom{n}{2i}$. We have

$$\begin{aligned}
a_n &= [x^n]\sum_n \sum_i 2i\binom{n}{2i}x^n \quad \text{(hint)} \\
&= [x^n]\sum_i 2i\sum_n \binom{n}{2i}x^n \quad \text{(change order)} \\
&= [x^n]\sum_i 2i\frac{x^{2i}}{(1-x)^{2i+1}} \quad \text{(standard identity)} \\
&= [x^n]\frac{2}{1-x}\sum_i iz^i \quad \left(z = \left(\frac{x}{1-x}\right)^2\right) \quad \text{(just rewriting)} \\
&= [x^n]\frac{2}{1-x}\frac{z}{(1-z)^2} \quad \text{(fiddling with derivatives)} \\
&= [x^n]\frac{2x^2-2x^3}{(1-2x)^2} \quad \text{(simplifying)} \\
&= [x^n]\left(2x^2\sum_i (i+1)2^i x^i - 2x^3\sum_i (i+1)2^i x^i\right) \quad \text{(known identity for } 1/(1-y)^2\text{)} \\
&= 2(n-1)2^{n-2} - 2(n-2)2^{n-3} \\
&= n2^{n-2}.
\end{aligned}$$

    Here's a direct argument: the sum is counting the number of even-sized committees-with-chair from n people, choosing the committee first. Choosing the chair first, the count is n times the number of odd-sized subsets of a set of size $n-1$; half of all possible subsets are odd-sized, leading to the 2^{n-2} term.

    The snake-oil method is longer and more intricate, and prone to mistakes when done by hand. On the other hand, it is a purely mechanical process and can easily be coded; it's a way of obtaining identities without "thought".

25. West 3.3.16

    Solution: $S(n,k)$ is the number of equivalence relations which have k non-empty parts; each one corresponds to $k!$ rankings (one for each ordering of the classes). So the number of rankings is

$$\sum_{k=1}^{n} k!S(n,k) = \sum_{k=1}^{n} k!\sum_{i=1}^{k}(-1)^{k-i}\frac{i^n}{i!(k-i)!}.$$

    Nothing much simpler than this is known.

26. West 3.3.37

Solution: Let $A(x)$ be the exponential generating function of permutations of n with only odd parts. We get each such permutation by splitting the label set $[n]$ into pieces and on each piece of size k putting an odd cycle of length k. So if $C(x)$ is the exponential generating function of $(c_k)_{k\geq 0}$ where c_k is the number of odd cycles of length k, we have, by the exponential formula,

$$A(x) = e^{C(x)}.$$

Since $c_k = 0$ if k is even and $(k-1)!$ if k is odd, we have

$$C(k) = 0 + x + 0x^2 + \frac{x^3}{3} + 0x^4 + \frac{x^5}{5} + \dots$$

and so

$$C'(k) = 1 + x^2 + x^4 + \dots = \frac{1}{1-x^2} = \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$$

and

$$C(k) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

so

$$A(x) = \sqrt{\frac{1+x}{1-x}}.$$

Now $B(x)$ is the exponential generating function of permutations of n with only odd parts, and an even number of them. This is the same as $A(x)$ except that for odd n, $B(x)$ has coefficient 0. The way to pick out the even coefficients is

$$2B(x) = A(x) + A(-x) = \sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}} = \frac{2}{\sqrt{1-x^2}},$$

as claimed.

We have $q = 0$ when n is odd, and when n is even (say $n = 2m$),

$$q = \frac{\binom{n}{n/2}}{2^n} = \frac{\binom{2m}{m}}{2^{2m}} = \frac{(2m)!}{m!m!2^{2m}} = \frac{1}{2^m m!}\frac{(2m)!}{2^m m!}.$$

On the other hand, p is the coefficient of x^n in $B(x)$ (note that $B(x)$ is the exponential generating function, so we have already normalized by the total number of permutations, $n!$). So for n odd, $p = 0$, and for $n = 2m$,

$$p = (-1)^{n/2}\binom{-1/2}{n/2} = (-1)^m\binom{-1/2}{m} = \frac{1.3.\dots(2m-3)(2m-1)}{2^m m!}.$$

It doesn't look very much like $p = q$, but a little manipulation gives

$$1.3.\dots(2m-3)(2m-1) = \frac{(2m)!}{2^m m!}$$

so that they are, in fact, equal.

27. West 4.1.19

    Solution: Imagine ordering the $2n$ courses, and assigning the first 2 courses to professor 1, etc. Dealing with the obvious overcount, this gives a total of

    $$\frac{(2n)!}{2^n}$$

    assignments.

    For the second part, let A_i be the set of assignments in the spring in which professor i repeats both his fall courses. We want to count the number of assignments which fall outside $A_1 \cup \cup A_n$. In the language of inclusion-exclusion, we want to calculate

    $$f(\emptyset) = \sum_S (-1)^{|S|} g(S)$$

    For S with $|S| = k$ (wlog, $S = 1, 2, \ldots, k$), $g(S) = \frac{(2n-2k)!}{2^{n-k}}$, so the number we require is

    $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2n-2k)!}{2^{n-k}}.$$

28. West 4.1.28

    Solution: We try to use inclusion-exclusion to count the number of injections from $[n]$ to $[n]$ (clearly $n!$). Let A_i be the number of functions from $[n]$ to $[n]$ that do not have i in the range. In the language of inclusion-exclusion, $f(\emptyset) = n!$ and $g(S) = (n - |S|)^n$, and so

    $$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n.$$

29. If G is a d-regular graph with n vertices, then nd must be even (since it's the sum of the degrees and so must be equal to twice the number of edges). Show that this trivial necessary condition on (n, d) for the existence of a d-regular graph with n vertices is also sufficient, by constructing a d-regular graph with n vertices for all $n \geq 1,\ 0 \leq d < n,\ nd$ even.

30. Petersen's Theorem states that a bridgeless cubic graph has a 1-factor.

    (a) Give an example of a cubic graph that *doesn't* have a 1-factor.

    (b) Give an example of a cubic graph with a bridge that *does* have a 1-factor.

31. Suppose that M is a matching in a bipartite graph that is not optimal (i.e., such that there is another matching M' with $|M'| > |M|$), then M has an augmenting path. (Note that M' and M need not be related in any way; so for example you can't assume that $M \subset M'$).