Discrete Mathematics, Spring 2009
Graph theory notation

David Galvin

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- **Graph**: a *graph* is a pair $G = (V, E)$ with V a set of *vertices* and E a set of *edges* — (unordered) pairs of vertices. The edge $e = \{x, y\}$ is often written $e = xy$; since edges are unordered, this is the same as yx. x and y are the *endpoints* of xy, and x and y are said to be *adjacent* or *neighbours*, written $x \sim y$ (or $x \leftrightarrow y$). They are also said to be *joined* by the edge xy. xy is *incident with* (or *to*) x (and y). By this definition of "graph", a pair of vertices can by joined by at most one edge, and no vertex can be joined to itself by an edge. Sometimes we relax these requirements, in which case a graph that satisfies these conditions is called *simple* and *loopless*.

- **Degrees**: the *degree* of a vertex is the number of edges it is an endpoint of, written $d_G(x)$ or $d(x)$. A graph is *regular* if all vertices have the same degree, and k-*regular* if all vertices have degree k. The minimum degree of a graph is denoted $\delta(G)$ and the maximum degree $\Delta(G)$. The *neighbourhood* of a vertex x is the set of vertices it is adjacent to, written $N_G(x)$ (or $N(x)$). A vertex is *isolated* if it has degree 0 or, equivalently, it has empty neighbourhood.

- **Size of a graph**: the number of vertices in a graph is denoted by $|V(G)|$ or n, and is sometimes called the *order*; the number of edges is denoted by $|E(G)|$ or m.

- **Paths and cycles**: a graph on n vertices is called a *path* (of length $n-1$) if the vertices can be labeled as $v_1, \ldots, v_n$ in such a way that $E = \{v_i v_{i+1} : i = 1, \ldots, n-1\}$. It is sometimes written $\langle v_1, \ldots, v_n \rangle$. It is called a *cycle* (of length n) if the vertices can be labeled as $v_1, \ldots, v_n$ in such a way that $E = \{v_i v_{i+1} : i = 1, \ldots, n-1\} \cup \{v_n v_1\}$. It is sometimes written $[v_1, \ldots, v_n]$. A cycle is *odd* if n is odd and *even* if n is even.

- **Subgraphs**: H is a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and H is a graph (i.e., the endpoints of the edges in $E(H)$ are all in $V(H)$). It is a *spanning* subgraph if $V(H) = V(G)$. It is an *induced* subgraph if $E(H) = \{xy : x \in H,\ y \in H,\ xy \in E(G)\}$. In this case, if $V(H) = S$ we write $H = G[S]$. Subgraphs are obtained from graphs by deleting vertices (and all incident edges), as well as deleting edges. Induced subgraphs are obtained by just deleting vertices (and all incident edges). The graph obtained from G by deleting the edge $e = xy$ is denoted $G - e$. The graph obtained from G by deleting the vertex x (and all incident edges) is denoted $G - x$.

- **Complete and empty graphs**: A graph is *complete* if every pair of vertices is adjacent, and empty if no pair of vertices is adjacent (so the edge set is empty). A *clique* in a graph is a set of vertices every pair of which is adjacent. A *stable* or *independent set* is a set of vertices no two of which are adjacent. These are clearly complementary notions. The *complement* of a graph G, written $\bar{G}$ is the graph of vertex set $V(G)$ with $xy \in E(\bar{G})$ if and only if $xy \notin E(G)$.

- k**-partite graphs**: A graph G is k-partite if it is possible to partition the vertex set into k pieces $V_1, \ldots, V_k$ in such a way that the induced subgraphs $G[V_1], \ldots, G[V_k]$ are all empty. The only 1-partite graph is the empty graph. Because the empty set is an empty graph, it follows that if a graph is k-partite for some k, then it is also ℓ-partite for all $\ell \geq k$.

- **Bipartite graphs**: A graph is *bipartite* if it is 2-partite, that is, the vertex set can be written as $V = X \cup Y$ in such a way that X and Y induce empty graphs (so all edges join a vertex of X and a vertex of Y). X and Y are call *bipartition* or *partition classes*. A *complete bipartite graph* is a bipartite graph with partition classes X and Y with every vertex of X joined to every vertex of Y.

- **Drawing**: A *drawing* of a graph $G = (V, E)$ is a collection of points in the plane labeled by V, with a smooth curve joining the point representing x to the point representing y whenever $xy \in E$ (notice that curves may intersect at places other than the points labeled by the vertices).

- **Adjacency and incidence matrices**: The *adjacency matrix* of a graph $G = (V, E)$ is the 0-1 matrix $A(G)$ whose rows and columns are labeled by vertices, with entry xy equal to 1 if and only if $xy \in E$. The *incidence matrix* is the 0-1 matrix $M(G)$ whose rows are labeled by vertices and whose columns are labeled by edges, with entry xe equal to 1 if and only if x is incident with e.

- **Isomorphism**: Graphs are labeled structures. But two graphs may have the same drawing, just with different labels, in which case for many purposes it is useful to consider them the same graph. The notion of isomorphism captures this. Two graphs G, H are *isomorphic* if there is a bijection $f : V(G) \to V(H)$ which preserves adjacency: $f(x)f(y) \in E(H)$ if and only if $xy \in E(G)$. An isomorphism induces a bijection from $E(G)$ to $E(H)$. Necessarily, isomorphic graphs must have the same size. If G and H are isomorphic we should write $G \cong H$ but often write $G = H$.

- **Isomorphism classes**: Isomorphism is an equivalence relation, and so partitions the set of graphs into equivalence classes. An *unlabeled graph* is an equivalence class of graphs. We often represent an unlabeled graph by a single example or representative drawing.

- **Named equivalence classes**: Some equivalence classes come up enough to merit special names. The name does not stand for a graph (although we often think of it that way); it stands for any graph that belongs to the described equivalence class.

- K_n: the complete graph with n vertices (K_3 is called the *triangle*)
- E_n: the empty graph with n vertices ($E_1 = K_1$)
- P_n: the path with n vertices
- C_n: the cycle with n vertices ($C_3 = K_3$; C_k is only defined for $k \geq 3$)
- $K_{p,q}$: the complete bipartite graph with p vertices in one class and q in the other ($K_{2,2} = C_4$; $K_{1,q}$ is called the *star*; $K_{1,3}$ is called the *claw*)

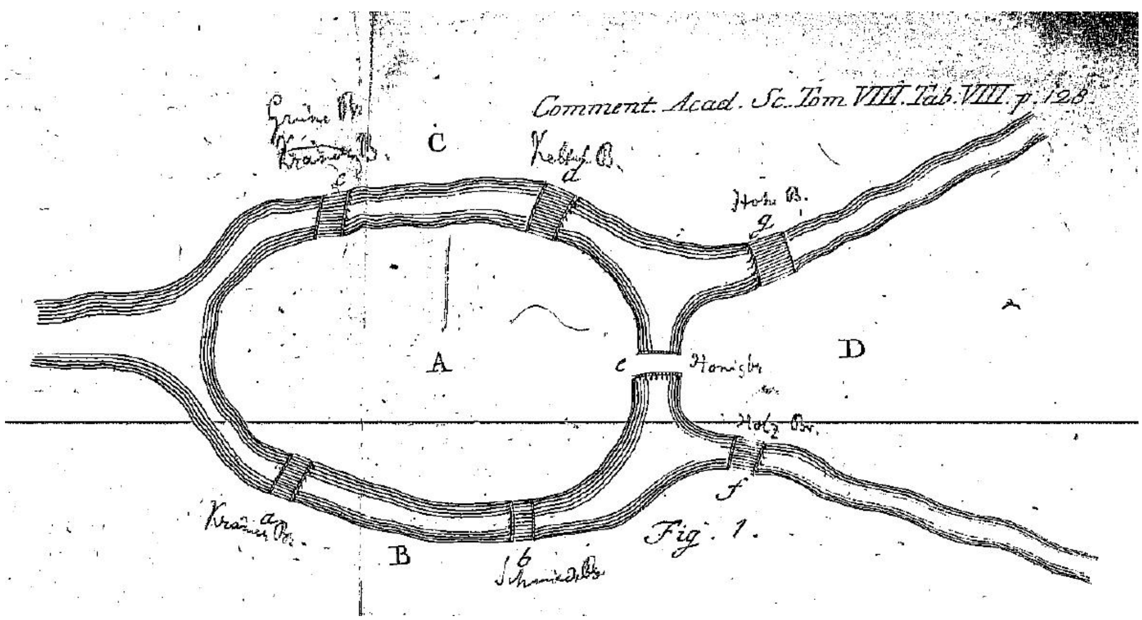
- **Subgraphs of unlabeled graphs**: If G and H are unlabeled graphs, and there are particular graphs G', H' in the equivalence classes of G and H with the property that H' is a subgraph of G', then every graph in the equivalence class of G contains a subgraph in the equivalence class of H. In this case it makes sense to say that H *is a subgraph of* G or G *contains a copy of* H.

- **Automorphisms**: An *automorphism* of a graph G is an isomorphisms from G to itself. The set of automorphisms form a group, the *automorphism group*. G is *vertex transitive* if for every $x, y \in V(G)$ there is an automorphism that sends x to y; it's *edge transitive* if for every $e, f \in E(G)$ there is an automorphism that sends e to f.

Moving around a graph

There are a variety of notions of "traversing" a graph. Here are pretty much all of them, defined informally.

- Path: $v_1, \ldots, v_n$ distinct, $v_i v_{i+1} \in E$
- Cycle: path with $v_n v_1 \in E$ also
- Walk: $v_1, e_1, \ldots, v_n$, $e_i = v_i v_{i+1}$ (edges and vertices may repeat)
- Closed walk: $v_n = v_1$ (I erroneously called this a circuit)
- Trial: A walk that doesn't repeat an edge
- Circuit or closed trail: a closed walk that doesn't repeat an edge
- Eulerian trial: a trial that visits every edge (necessarily only once)
- Eulerian circuit: a circuit that visits every edge (necessarily only once)
- Hamilton path: a path that visits each vertex (necessarily only once)
- Hamilton cycle: a cycle that visits each vertex (necessarily only once)



A map of the bridges of Königsberg, circa 1736, taken from Euler's paper