The principle of inclusion-exclusion

Math 40210, Fall 2012

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Inclusion-Exclusion

Given a set of N objects, and a set of r properties $1, \ldots, r$.

- $N(i)$ of the objects have (at least) property i, for each i
- $N(ij)$ have (at least) properties i and j, for each $i < j$
- $N(i_1 i_2 \ldots i_k)$ have (at least) properties $i_1, i_2, \ldots, i_k$, for each $i_1 < i_2 < \ldots < i_k$

The number of objects with none of the properties is given by

$$\begin{aligned} N_0 &= N - \sum_i N(i) + \sum_{i<j} N(ij) - \ldots \\ &+ (-1)^k \sum_{i_1 < i_2 < \ldots < i_k} N(i_1 i_2 \ldots i_k) \ldots \\ &+ (-1)^r N(123 \ldots r) \end{aligned}$$

Alternate formulation

$A_1, \ldots, A_r$ subsets of some set Ω

The size of the union of the A_i, and its complement, are given by

$$
\begin{aligned}
|A_1 \cup \ldots A_r| &= \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \ldots \\
&\quad + (-1)^{k+1} \sum_{i_1 < i_2 < \ldots < i_k} |A_{i_1} \cap \ldots \cap A_{i_k}| \ldots \\
&\quad + (-1)^{r+1} |A_1 \cap \ldots \cap A_r|
\end{aligned}
$$

$$
\begin{aligned}
|(A_1 \cup \ldots A_r)^c| &= |\Omega| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \ldots \\
&\quad + (-1)^{k} \sum_{i_1 < i_2 < \ldots < i_k} |A_{i_1} \cap \ldots \cap A_{i_k}| \ldots \\
&\quad + (-1)^{r} |A_1 \cap \ldots \cap A_r|
\end{aligned}
$$