The basic rules of counting

Math 40210, Fall 2012

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Basic counting rule 1 — The sum rule

- **Sum rule 1**: if an experiment can proceed in one of two ways, with
  - n_1 outcomes for the first way, and
  - n_2 outcomes for the second,

  then the total number of outcomes for the experiment is

  $$n_1 + n_2$$

  Example: Movie *or* dinner? $\#$(screens) $+$ $\#$(restaurants)

- **Sum rule 2**: if an experiment can proceed in one of m ways, with
  - n_1 outcomes for the first way,
  - n_2 outcomes for the second, ..., and
  - n_m outcomes for the mth,

  then the total number of outcomes for the experiment is

  $$n_1 + n_2 + \ldots + n_m$$

Basic counting rule 2 — The product rule

- **Product rule 1**: if an experiment is performed in two stages, with
  - n_1 outcomes for the first stage, and
  - n_2 outcomes for the second, *REGARDLESS OF FIRST*,

  then the total number of outcomes for the experiment is

  $$n_1 n_2$$

  Example: Movie *and* dinner? $\#$(screens) $\times$ $\#$(restaurants)

- **Product rule 2**: if an experiment is performed in m stages, with
  - n_1 outcomes for the first stage, and
  - n_2 outcomes for the second, *REGARDLESS OF FIRST*, . . ., and
  - n_m outcomes for the mth, *REGARDLESS OF ALL PREVIOUS*,

  then the total number of outcomes for the experiment is

  $$n_1 n_2 \ldots n_m$$

How many ways to order n distinct items?

$$n(n-1)\ldots 3.2.1 = n!$$

Example: Outcomes in race with 8 runners?
$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

Selecting k items from n, WITHOUT REPLACEMENT

- **Order matters**:

$$n(n-1)\ldots(n-(k-1)) = n^{\underline{k}} = \frac{n!}{(n-k)!}$$

Example: 1st, 2nd and 3th in race with 8 runners? $8 \times 7 \times 6 = 336$

- **Order doesn't matter**:

$$\frac{n^{\underline{k}}}{k!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Example: Top three in race with eight runners? $\binom{8}{3} = 56$

Basic counting rule 3 — The overcount rule

- If x is an initial count of some set of objects, and each object you want to count appears y times in x, then the correct count is

$$x/y$$

Some simple examples

- How many ways can 10 people form a committee of 6, with a chair, a secretary a treasurer and 3 general members?

$$\binom{10}{6}6.5.4 = 25200$$

- What if John and Pat will not serve together, and Pat will only serve if Ellen is the chair?
  First consider committees with John (so without Pat), then those with Pat (so also with Ellen as chair, and without John), and then those without both John and Pat

$$\binom{8}{5}6.5.4 + \binom{7}{4}5.4 + \binom{8}{6}6.5.4 = 10780$$

- How many 9 digit numbers can be formed with four 1's, three 2's and two 3's?

$$\frac{9!}{4!3!2!} = 1260$$

Some more examples

- A bridge hand consists of 13 cards from a deck of 52. How many are there?

$$\binom{52}{13} = 635013559600$$

- How many have no Aces, Tens or picture cards?

$$\binom{32}{13} = 347373600 \text{ (so about 1 in 2000)}$$

- How many have two 4-of-a-kind, one pair, and no other repeats?

$$\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{10}{3}4^3 = 6589440 \text{ (so about 1 in 100000)}$$