Some problems leading to Catalan numbers

Math 30530, Fall 2012

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The Catalan recurrence, and some values

$$
\begin{aligned}
c_0 &= 1 \\
c_n &= c_0 c_{n-1} + c_1 c_{n-2} + \ldots + c_{n-1} c_0 \\
&= \sum_{k=0}^{n-1} c_k c_{n-1-k} \text{ for } n \geq 1
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= 1 \\
c_1 &= 1 \\
c_2 &= 2 \\
c_3 &= 5 \\
c_4 &= 14 \\
c_5 &= 42
\end{aligned}
$$

Some problems leading to Catalan numbers

- **Handshakes**: c_n counts number of ways that $2n$ people in a circle can pair off to shake hands, with no crossing hands
- **One-sided tied games**: c_n counts number of ways the Cubs and White Sox can play to an n-n tie , in which the Cubs never lead (games considered by the order in which the runs are scored)
- **Triangulations**: c_n counts the number of different ways that a convex $(n+2)$-gon can be fully triangulated
- **Trees**: c_n counts the number of full binary trees with $n+1$ leaves. (Start with a root. Each vertex either has two children (right and left), or no children.)
- **Tiling stairs**: c_n counts the number of ways of tiling a height n staircase with exactly n rectangles. (The height n staircase is the set of 1 by 1 boxes whose top right points are the points (i, j) with $i, j \geq 1$ and $i + j \leq n + 1$.)
- R. Stanley, *Enumerative Combinatorics*, has an exercise that gives 66 different counting problems, all solved by Catalan numbers; an addendum on his website gives 136 more!