

Basic Combinatorics (Math 40210) Sec 01, Fall 2012, Quiz 4

Solutions

October 31, 2012

This question is about the binomial coefficient identity

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.$$

valid for $n \geq k \geq 1$.

1. Prove the identity using the algebraic representation of $\binom{n}{k}$.

Solution:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n}{k} \binom{n-1}{k-1}.$$

2. Prove the identity combinatorially, by showing that $k\binom{n}{k}$ and $n\binom{n-1}{k-1}$ count the same thing.

Solution: Imagine choosing a team of k people from among a pool of n people, where one person on the team is designated captain. If you choose the team first, and then choose the captain from among the team, then there are

$$\binom{n}{k} \binom{k}{1} = k \binom{n}{k}$$

total outcomes. If you choose the captain first (one person from among the n) and then, from the remaining $n-1$ people, choose the rest of the team ($k-1$ people), then there are

$$\binom{n}{1} \binom{n-1}{k-1} = n \binom{n-1}{k-1}$$

total outcomes. Both experiments lead to the same set of outcomes, so the two counts must be the same.