Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 7 — Solutions

- **2.1 1a**: 53 choice for initial character, 63 for all the rest, so (53)(63)(63)(63)(63) in total.
- **2.1 1b**: 53 with one character, 53×63 with two characters, so $(53) + (53)(63) + (53)(63)(63) + (53)(63)(63)(63) + (53)(63)(63)(63)(63)$ in total.
- **2.1 1c**: 53 with one character, 53 with two characters (first character determines second), 53×63 with three characters (first character determines third), 53×63 with four characters (first two characters determines third and fourth), $53 \times 63 \times 63$ with five characters (first two characters determine fourth and fifth), so $(53) + (53) + (53)(63) + (53)(63) + (53)(63)(63)$ in total.
- **2.1 3a**: 30!
- **2.1 3b**: (14)(13)(12)
- **2.1 3c**: $\binom{15}{8} \times \binom{15}{8}$
- **2.1 3d**: In the western division there are in total 45 centers from which three must be chosen, 60 guards from which four must be chosen, and 75 forwards from which five must be chosen, leading to a total of $\binom{45}{3}\binom{60}{4}\binom{75}{5}$.
- **2.1 7**: There are $1+10+\binom{10}{2}+\binom{10}{3}+\binom{10}{4} = 386$ ways to choose up to four candidates for city council (this includes choosing no-one). There are $1+8+(8\times 7)+(8\times 7\times 6) = 401$ ways to rank up to three candidates for the school board (this includes ranking no-one). There are $3^5 = 243$ choices for the ballot measure (each one is either accepted, rejected or ignored; this includes accepting/rejecting none). The total number of different ballots is then
$$386 \times 401 \times 243.$$
- **2.1 11**: An n which is a positive integer divisor of N has prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where each α_i satisfies $0 \leq \alpha_i \leq n_i$. So there are $n_i + 1$ choices for each α_i, leading to $(n_1+1)(n_2+1)\dots(n_m+1)$ distinct positive integer divisors of N.
- **2.2 2**: An algebraic proof is easy:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n}{k}\binom{n-1}{k-1}.$$

For a combinatorial proof, notice that multiplying through by k one gets

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

This is exactly the identity from Quiz 4. It's often called the committee-chair identity. See the quiz solutions for a counting proof.

- **2.2 3**: We have

$$\begin{aligned}
\binom{n}{k}\binom{k}{m} &= \frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} \\
&= \frac{n!}{(n-k)!}\frac{1}{m!(k-m)!} \\
&= \frac{n!}{m!}\frac{1}{(k-m)!(n-k)!} \\
&= \frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!(n-k)!} \\
&= \binom{n}{m}\binom{n-m}{k-m}.
\end{aligned}$$

- **2.2 4**: Either he first selects the k paintings to display from the n, and then selects the m paintings from the k to display prominently, leading to a count of

$$\binom{n}{k}\binom{k}{m},$$

or he first chooses the m paintings from the n to display prominently, then chooses the remaining $k-m$ from the remaining $n-m$ to also display (but less prominently), leading to a count of

$$\binom{n}{m}\binom{n-m}{k-m}.$$

Notice that this is a generalization of the identity from Quiz 4 (which is the case $m=1$).

- **2.2 5**: How many ways to choose n objects for $m+n+1$? $\binom{m+n+1}{n}$. That's the direct way to count. Here's a less direct way: any selection of n items from $m+n+1$, say the $m+n+1$ items are $a_1, a_2, \ldots, a_{n+m+1}$, can be obtained by first selecting a consecutive block starting from a_1, then not selecting the first element after the block, and then selecting the rest of the elements from beyond the block. If the initial block chosen is a_1 through a_n, then the remaining 0 elements have to be chosen from a_{n+2} through a_{n+m+1}, a list of length m; so $\binom{m}{0} = \binom{m+0}{0}$ ways. If the initial block chosen is a_1 through a_{n-1}, then the remaining 1 element has to be chosen from a_{n+1} through a_{n+m+1}, a list of length $m+1$; so $\binom{m+1}{1}$ ways. If the initial block chosen is a_1 through a_{n-1}, then the remaining 2 elements have to be chosen from a_n through a_{n+m+1}, a list of length $m+2$; so $\binom{m+2}{2}$ ways. We keep going like this until we get to the initial block being just a_1, leaving the remaining $n-1$ elements to be chosen from a_3 through a_{n+m+1}, a list of length $m+(n-1)$; so $\binom{m+(n-1)}{n-1}$ ways; and finally, we have the initial block being empty, leaving all n elements to be chosen from a_2 through a_{n+m+1}, a list of length $m+n$; so $\binom{m+n}{n}$ ways. So in total there are

$$\sum_{k=0}^{n} \binom{m+k}{k}$$

ways to choose. The right- and left-hand sides are counting the same thing, so they are equal.

- **2.2 7d**: This is a tricky one! Without using $\binom{n}{k} = \binom{n}{n-k}$, I know of no way to approach this (no algebraic or inductive proof, for example). Using $\binom{n}{k} = \binom{n}{n-k}$, we have

  $$\sum_k \binom{n}{k}^2 = \sum_k \binom{n}{k}\binom{n}{n-k}.$$

  The right hand-side is counting the number of ways of selecting a set of size n from a set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ of size $2n$, by first deciding how many of the n comes from the a_i's (k of them, leading to a count of $\binom{n}{k}$), forcing the remainder to comes from the b_i's ($n-k$ of them, leading to a count of $\binom{n}{n-k}$). But by a direct count, we get that this is just $\binom{2n}{n}$. (Notice that this is an example of the vandermonde convolution from page 142, with $m = \ell = n$ in the displayed equation above (2.11)). In summary:

  $$\sum_k \binom{n}{k}^2 = \binom{2n}{n}.$$

- **2.2 7e**: If $n = 0$ and m is negative, then the sum is 0 (it is empty). If $n = 0$ and $m \geq 0$, then the sum is 1. That deals with $n = 0$; so from now on we assume $n \geq 1$.

  For $n \geq 1$, if $m < 0$, then the sum is 0 (it is empty). For $m = 0$, there's just one term, and the sum is 1. For $m \geq n$, the sum is the same as if we stopped at n, so it's 0, as we proved in class. So the remaining (and most interesting) cases are $n \geq 2$ and $1 \leq m \leq n-1$.

  A little experimentation with Pascal's triangle suggests that in this range:

  $$\sum_{k \leq m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

  For each fixed $n \geq 2$, we prove this by induction on m, with the case $m = 0$ trivial. For $m > 0$ we have

  $$\begin{aligned}
  \sum_{k \leq m} (-1)^k \binom{n}{k} &= \binom{n}{m} + \sum_{k \leq m-1} (-1)^k \binom{n}{k} \\
  &= (-1)^m \binom{n}{m} + (-1)^{m-1} \binom{n-1}{m-1} \\
  &= (-1)^m \left(\binom{n}{m} - \binom{n-1}{m-1} \right) \\
  &= (-1)^m \binom{n-1}{m}
  \end{aligned}$$

the second equality using the induction hypothesis and the last equality using Pascal's identity.

- **2.2 8** (just for falling powers): I don't know of an easy way to do this algebraically or by induction. If we allow ourselves to only prove the result for x, y positive integers, then there are two easy approaches:

  First, we can write

$$\begin{aligned}\binom{n}{k}x^{\underline{k}}y^{\underline{n-k}} &= \frac{n!}{k!(n-k)!}x^{\underline{k}}y^{\underline{n-k}} \\ &= n!\frac{x^{\underline{k}}}{k!}\frac{y^{\underline{(n-k)}}}{(n-k)!} \\ &= n!\binom{x}{k}\binom{y}{n-k}.\end{aligned}$$

  So, by vandermonde's convolution, we have

$$\sum_k \binom{n}{k}x^{\underline{k}}y^{\underline{n-k}} = \sum_k n!\binom{x}{k}\binom{y}{n-k} = n!\sum_k \binom{x}{k}\binom{y}{n-k} = n!\binom{x+y}{n}.$$

  But also

$$(x+y)^{\underline{n}} = n!\binom{x+y}{n}.$$

  So we have the identity.

  Here's another, more combinatorial way: the left hand side directly counts the number of ways of taking n elements from a set of size $x+y$, say $\{a_1, \ldots, a_x, b_1, \ldots, b_y\}$, and arranging the n elements in order. Another way to do this is to select k elements (k running from 0 to n) from $\{a_1, \ldots, a_x\}$ and arrange them in order ($x^{\underline{k}}$ ways to do this), take $n-k$ elements from $\{b_1, \ldots, b_y\}$ and arrange them in order ($y^{\underline{(n-k)}}$ ways to do this), and then merge the two ordered sets to get an ordered list of n elements from the full set of a's and b's ($\binom{n}{k}$ ways to do this - just choosing the k slots into which the a's go). So the right hand side also counts the number of ways of taking n elements from a set of size $x+y$ and arranging them in order.

- **2.3 2** (it should be clarified that the steps must always be taken in a *positive* direction: you can go from (x, y, z) to any of $(x+1, y, z)$, $(x, y+1, z)$ or $(x, y, z+1)$, but not for example to $(x-1, y, z)$.): In order to reach (a, b, c) from $(0, 0, 0)$ taking steps parallel to (and in the same direct as) $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, we need to take exactly $a+b+c$ steps. a of these steps must be steps of the form $(1, 0, 0)$, b of them must be of the form $(0, 1, 0)$, and c of them must be of the form $(0, 0, 1)$. So we completely determine a path by partitioning the set $\{1, \ldots, a+b+c\}$ into three classes, class 1 of size a, class 2 of size b and class 3 of size c, with i falling into class 1 indicating that the ith step is of the form $(1, 0, 0)$, etc.. There are exactly $\binom{a+b+c}{a,b,c}$ such partitions.

- **2.3 5**: From a set of size n, first choose a subset of size k, and then choose an arbitrary subset from the elements not chosen for the set of sized k. There are $\binom{n}{k}$ ways to choose the first subsets, and subsequently 2^{n-k} ways to choose the second subset, and so $2^{n-k}\binom{n}{k}$ ways in all to do the selection.

What you've done is divide the set of size n into 3 classes: class one is of size k (the initial subset), class 2 is of some variable size j (the second subset) and class 3 is of size $n-k-j$. For each possible size j of the second set, there are (by our definition of the multinomial coefficient), a total of $\binom{n}{k,j,n-j-k}$ ways to create the 3 classes. Since j is variable, the total number of ways of creating the 3 classes is

$$\sum_j \binom{n}{k,j,n-j-k} = \sum_j \binom{n}{j,k,n-j-k}.$$

The right- and left-hand sides are counting the same thing, so they are equal.

Here's an algebraic proof, using the binomial theorem:

$$\begin{aligned}
\sum_j \binom{n}{k,j,n-j-k} &= \sum_j \frac{n!}{k!j!(n-j-k)!} \\
&= \sum_j \frac{n!(n-k)!}{(n-k)!k!j!(n-j-k)!} \\
&= \sum_j \binom{n}{k}\binom{n-k}{j} \\
&= \binom{n}{k}\sum_j \binom{n-k}{j} \\
&= \binom{n}{k}2^{n-k}.
\end{aligned}$$

- **2.3 7**: The left hand side counts the number of ways of partitioning a set of size $m+n$, say $\{a_1,\ldots,a_m\}\cup\{b_1,\ldots,b_n\}$ into three classes, the first of size a, the second of size b and the third of size c. The count is direct.

  Another (indirect) way to count the same thing is to first decide how many of the a_i's go into class 1 (say α of them), how many of the a_i's go into class 2 (say β of them), and how many of the a_i's go into class 3 (say γ of them), then count the number of partitions that actually achieve this split (the summand of the right hand side counts exactly this: if α of the a_i's go into class 1, then $a-\alpha$ of the b_i's must, etc.), then sum this quantity over all possible choices of α, β and γ (for which the only constraint is $\alpha+\beta+\gamma=m$, since all of the a_i's must go into *some* class). This is exactly the right-hand side.

  NB: I'm not vouching for the 100% accuracy of the numbers from here on - please let me know if you spot errors!

- **2.3 9d**: We have 6 A's, 2 K's, 2 L's, 2 S's, 1 N, and 1 U.

  - $r=3$: Total 181.

    - 1 choice for word type xxx, each with 1 anagram;
    - 20 choices for word type xxy, each with 3 anagrams;
    - 20 choices for word type xyz, each with 6 anagrams.
  - $r = 4$: Total 897.
    - 1 choice for word type $xxxx$, each with 1 anagram;
    - 5 choices for word type $xxxy$, each with 4 anagrams;
    - 6 choices for word type $xxyy$, each with 6 anagrams;
    - 40 choices for word type $xxyz$, each with 12 anagrams;
    - 15 choices for word type $xyzw$, each with 24 anagrams.
  - $r = 14$:
$$\binom{14}{6,2,2,2,1,1} = \frac{14!}{6!2!2!2!}$$
- **2.3 9e**: We have 5 A's, 4 L's, and 3 W's.
  - $r = 4$: Total 80
    - 2 choices for word type $xxxx$, each with 1 anagram;
    - 6 choices for word type $xxxy$, each with 4 anagrams;
    - 3 choices for word type $xxyy$, each with 6 anagrams;
    - 3 choices for word type $xxyz$, each with 12 anagrams;
    - 0 choices for word type $xyzw$, each with 24 anagrams.
  - $r = 5$: Total 231
    - 1 choice for word type $xxxxx$, each with 1 anagram;
    - 4 choices for word type $xxxxy$, each with 5 anagrams;
    - 6 choices for word type $xxxyy$, each with 10 anagrams;
    - 3 choices for word type $xxxyz$, each with 20 anagrams;
    - 3 choices for word type $xxyyz$, each with 30 anagrams;
    - 0 choices for all other word types.
  - $r = 12$:
$$\binom{12}{5,4,3} = \frac{12!}{5!4!3!}$$