Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 5 — Solutions

- **1.5.2 1**: $n = 24$ and $2q = \sum_v \deg(v) = 24 \times 3 = 72$, so $q = 36$, meaning that in any planar representation we must have $r = 2 + q - n = 2 + 36 - 24 = 14$.

- **1.5.2 4**: If G is a tree, then $q = n - 1 \leq 2n - 4$ (because $n \geq 3$). So assume it is not a tree. Let's look at $\sum_{\text{regions } R} b(R)$ (note that since we are assuming G is not a tree, there are at least 2 regions). Since G has no triangles, this is at least $4r = 4(2 + q - n) = 8 + 4q - 4n$ (this is using Euler's formula, since G is connected); but since each edge contributes as a bounding edge to at most two regions, the sum is at most $2q$. So we get $8 + 4q - 4n \leq 2q$, or, rearranging, $2q \leq 4n - 8$, or $q \leq 2q - 4$.

- **1.5.2 6**: Imagine k disjoint boxes in $\mathbb{R}^2$, arranged in a row. Draw a planar representation of G with each component sitting inside a different box. Add $k - 1$ edges to the graph, joining a vertex in the first box to one in the second, a vertex in the second box to one in the third, and so on, making sure that no crossing are created (see the figure 1 of the figures page). We now have a connected planar graph with n vertices, $q + k - 1$ edges and r faces, where n, q and r are the numbers of vertices, edges and faces of the original graph. (Note that we have not created any new faces by adding the edges; these edges travel through the infinite outer face, but don't disconnect it). Applying Euler's formula to this new graph, we get $n - (q + k - 1) + r = 2$ or $n - q + r = k + 1$.

- **1.5.2 7**: Suppose G has n vertices and m edges. If G is not planar, we are done. If it is planar, then we want to say that $\overline{G}$ is *not* planar. From a theorem in class, it is enough to show that m', the number of edges in $\overline{G}$, satisfies $m' > 3n - 6$. Now since G is planar, the same theorem tells us that $m \leq 3n - 6$, and the number of edges in $\overline{G}$ is $m' = (n^2 - n)/2 - m$ (since there are $(n^2 - n)/2$ possible edges, each of which must be in one of G, $\overline{G}$). Combining these observations, we see that $m' \geq (n^2 - n)/2 - 3n + 6$. So its enough to show that $(n^2 - n)/2 - 3n + 6 > 3n - 6$, or $n^2 > 13n - 24$. This is true for all $n \geq 11$.

- **1.5.2 9**: $K_{3,3}$ provides a counterexample to the converse: $q = 9$, $n = 6$, so $q \leq 3n - 6$ (and $n \geq 3$); but $K_{3,3}$ is not planar.

- **1.5.4 2**: See figure 2 of the figures page, for an example of how to find a subdivision of $K_{3,3}$ as a subgraph of the Petersen graph.

- **1.6.1 1a**: The stump K_1 clearly has chromatic number 1. All other trees are bipartite (no cycles, so in particular no odd cycles), and so can be 2-colored, meaning that their chromatic number is at most 2. But all trees except the stump have at least one edge, and so need at least two colors (one for each endvertex of any particular edge) for a proper coloring, meaning that these trees have chromatic number at least 2. Combining these two observations, we see that all trees that are not the stump have chromatic number exactly 2.

- **1.6.1 1c**: Once we use a color in one of the partite sets, we cannot use it in any of the others (since every vertex in one partite set is adjacent to *every* vertex in every other partite set). So we need at least t colors in a proper coloring (one for each partite class). But t colors are enough: assign color i to *every* vertex in the ith partite set, for each $i = 1, \ldots, t$, to get a proper t-coloring. This shows that $\chi(K_{r_1,\ldots,r_t}) = t$. (I'm assuming here, as is conventional, that each $r_i > 0$.)

- **1.6.1 1d**: The Petersen graph has a 5-cycle, and so we need at least 3 colors just to color that. But 3 colors is enough to properly color the whole graph, as we see in figure 3 of the figures page. This shows that the chromatic number of the Petersen graph is 3.

- **1.6.1 1e**: The chromatic number of the Birkhoff Diamond is 3. It's at least 3 because it has a K_3; the coloring in figure 4 of the figures page shows that it is at most 3.

- **1.6.1 2**: We draw a graph on vertex set $\{c_1, \ldots, C_7\}$, with two vertices adjacent if the corresponding committees have a member in common. We need to determine the chromatic number of this graph, in order to see what is the minimum number of timeslots needed to allow all committees to meet. The graph is shown in figure 5 of the figures page, together with a 3-coloring that shows that the chromatic number is 3 (because the graph has triangles, the chromatic number must be at least 3). So three times slots are needed.

- **1.6.1 4**: Adding an edge cannot *decrease* the chromatic number: if there is a k-coloring after the edge $e = uv$ has been added, it is still a k-coloring when the edge is removed. So now, let G have chromatic number k, and let G' be the graph with edge e added. Let K be a proper k coloring of G. If $K(u) \neq K(v)$, then K is also a proper k-coloring of G', and so $\chi(G') = \chi(G)$. If $K(u) = K(v)$, then K is *not* a proper k-coloring of G', but we can use it to get a proper $(k+1)$-coloring K': let $K'(x) = K(x)$ for all $x \neq u$, and let $K'(u) = k + 1$. This shows that $\chi(G') \leq \chi(G) + 1$ (although in this case, we do not know which of $\chi(G') = \chi(G)$, $\chi(G') = \chi(G) + 1$ is true without further investigation). But in either case, the chromatic number increases by at most 1 by adding an edge.

- **1.6.1 6a**: A 1-critical graph has chromatic number 1, so must be an empty graph E_n. Which n? If $n > 1$, then on the removal of any vertex, we still have an empty graph with chromatic number 1, and so the graph is not 1-critical. But if $n = 1$, when we remove the only vertex we get a graph which has no vertices, and so has chromatic number 0. So the only 1-critical graph is $E_1 = K_1$, the stump.

A 2-critical graph has chromatic number 2, so must be a bipartite graph with at least one edge. On deleting any vertex, we must have an empty graph (the only graphs with chromatic number 1). So *every* vertex must be adjacent to *every* edge. The only graph with this property is K_2, so this is the only 2-critical graph.

- **1.6.1 6b**: K_3 is 3-critical.

- **1.6.1 6c**: Suppose that G is k-critical, but *not* connected. At least one component, say C_1, of G must have chromatic number k (if not, we could color each component and so the whole graph with $k-1$ colors). Let v be a vertex *not* in C_1. After deleting v, C_1 still needs k colors to be properly colored, contradicting the fact that G is critical. So G must be connected

- **1.6.1 6d**: Suppose that G is k-critical, but has a vertex of degree $k-2$ or less, say v. By criticality, we have a $k-1$ coloring of $G-v$. But in any such coloring, at most $k-2$ colors appear on the neighbors of v, so we can add v back and give it a color from $\{1,\ldots,k-1\}$ different from the color of its neighbors, leading to a $(k-1)$-coloring of G, a contradiction. So G has minimum degree at least $k-1$.

- **1.6.1 6e**: Let G be 3-critical. Since it has chromatic number 3, it cannot be bipartite, and so must have some odd cycles. On the deletion of any vertex, the graph must be bipartite with at least one edge, and so *every* vertex must belong to *every* cycle. If G has even two odd cycles, then this is impossible (there must be at least one vertex on one odd cycle, but not on the other). So G is a connected graph, with exactly one odd cycle, and every vertex of the graph must be on this cycle. One possibility: G is an odd cycle.

  If G is *not* and odd cycle, then it is an odd cycle with some extra edges. But however we add an edge to an odd cycle, we must create two different odd cycles. To prove this, let $C = v_1v_2\ldots v_kv_1$ be the odd cycle. If we add the edge v_1v_ℓ for some odd ℓ, then $v_1\ldots v_\ell v_1$ is an odd cycle different from C. If we add the edge v_1v_ℓ for some even ℓ, then $v_1v_ell\ldots v_kv_1$ is an odd cycle different from C. (See figure 6 of the figures page). So either way, we create a new odd cycle, which is not allowed, since we have argued that all 3-critical graphs are allowed to have only one odd cycle.

  This means that the only 3-critical graphs are the odd cycles. (I'm not sure how part d) helps here. Part c) certainly does.)

- **1.6.2 1**: This bound is too good to be true, so there must be a counterexample. The simplest one is the graph on three vertices with one edge (see figure 7 of the figures page). This graph has average degree 2/3, so the right-hand side of the proposed inequality is $1+2/3$; but the chromatic number of the graph is 2, which is larger than 12/3. For a connected counterexample, take K_4 with an extra edge hanging off one vertex (see figure 8 of the figures page). This graph has chromatic number 4, but 1 plus the average degree is only 34/5. For an infinite family of counterexamples, take K_n together with a single extra vertex, not adjacent to anything, for $n \geq 2$; for an infinite family of connected counterexamples, take K_n with an extra edge hanging off one vertex, for $n \geq 4$.

- **1.6.2 4**: We prove this by induction on $\tau(G)$. If $\tau(G) = 1$, then G is an empty graph E_n, so $\chi(G) = 1$. So now consider what happens when $\tau(G) = k > 1$. Find a maximum independent set I, and give everything in that set color k. We want to say that when we remove I from G (to get G'), we can color G' using $k-1$ colors. If we could show that the value of τ drops (by at least 1) when we remove a maximum independent set, then we would have $\tau(G') \leq k-1$, so by induction we can indeed color G' with at most $k-1$ colors and so G with at most k.

  So here's what we need to show: if I is a maximum independent set in G, then $\tau(G-I) \leq \tau(G)-1$. If this was *not* true, then there would be a path of maximum length in G that misses I. Consider an endvertex v of that path. If v is adjacent to anything in I, then the path can be extended by added a neighbor of v in I, a contradiction (of the path being as long as possible). So v is not adjacent to anything in I, and that means I can be extended to the larger independent set $I \cup \{v\}$, also a contradiction (this time of the independent set being maximal). Since either way we get a contradiction, it must be that $\tau(G-I) \leq \tau(G) - 1$.

- **1.6.2 6**: First the lower bound: let the chromatic number be k, and let $S_1, \ldots, S_k$ be the color classes of a particular k-coloring (that is, S_i is the set of vertices colored i in the coloring). Each S_i is an independent set in G, so $|S_i| \leq \alpha(G)$. But also, the S_i's cover all of G, so $\sum_{i=1}^{k} |S_i| = n$. Combining, we get $n = \sum_{i=1}^{k} |S_i| \leq \sum_{i=1}^{k} \alpha(G) = k\alpha(G)$, so $k \geq n/\alpha(G)$, as required.

  Now the upper bound: we show that there is coloring that uses $n - \alpha(G) + 1$ colors. Simply use color 1 on every vertex of an independent set of size $\alpha(G)$, and use a different color for each of the remaining $n - \alpha(G)$ vertices, for a total of $n - \alpha(G) + 1$.

- **1.6.2 8a**: We use $\chi(G) \geq n/\alpha(G)$ and $\chi(\overline{G}) \geq \omega(\overline{G})$. Multiplying, we get $\chi(G)\chi(\overline{G}) \geq n\omega(\overline{G})/\alpha(G)$. But an independent set in G is exactly a clique in $\overline{G}$, so $\omega(\overline{G}) = \alpha(G)$. Thus we get $\chi(G)\chi(\overline{G}) \geq n$.

- **1.6.2 8b**: From part a), we know that $2\sqrt{n} \leq 2\sqrt{\chi(G)\chi(\overline{G})}$. Thus it is enough to show that for any positive numbers a, b, we have $2\sqrt{ab} \leq a + b$. This is equivalent to $4ab \leq (a+b)^2 = a^2 + 2ab + b^2$, which is the same as $0 \leq a^2 - 2ab + b^2 = (a-b)^2$, which is true since the square of any number, positive or negative, is non-negative.

- **1.6.2 additional**: The following is a simple example: start with $K_{n/2}$ on vertex set $v_1, \ldots, v_{n/2}$, and then add vertices $u_1, \ldots, u_{n/2}$ with each u_i joined only to v_i. This graph G has $\alpha = n/2$ (easy to see), as well as $\chi(G) = n/2$, as required. To see that $\chi(G) = n/2$, note that $\omega(G) = n/2$ (easy), so $\chi(G) \geq n/2$, but also there is a coloring using $n/2$ colors - color each v_i with color i, and color u_1 with color 2, u_2 with color 3, $\ldots$, u_{n-1} with color n and u_n with color 1 (a coloring which does *not* work when $n = 2$) – showing that $\chi(G) = n/2$.

- **1.6.3 2**: Four colors are needed to color a map of South America. To see this, notice that Paraguay, Brazil, Bolivia and Argentina are all mutually adjacent, and so form a K_4 in the planar graph corresponding to the map. To see an example of a 4-coloring, see `http://www.infoplease.com/atlas/southamerica.html`.