Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 4 — Solutions

- **1.4.2 2**: One possible implementation:
  - Start with $abcgfjiea$
  - From edge cd build, using previously unmarked edges: $cdhlponminjkghc$
  - Patch first two together: ab**cdhlponminjkghc**$gfjiea$
  - From edge kl build, using previously unmarked edges: $klok$
  - Patch into what we have so far: $abcdhlponminj$**klok**$ghcgfjiea$
  - From edge eb build, using previously unmarked edges: $ebfb$
  - Patch into what we have so far: $abcdhlponminjklokghcgfji$**ebfb**a
  - No more edges! Have Euler circuit $abcdhlponminjklokghcgfjiebfba$

- **1.4.2 4**: Suppose G is connected and has an Euler trail. Either: the trail is a circuit, in which we know (from a theorem) that all degrees are even. Or: the trail is not a circuit. Suppose in this case that it starts at a and ends at $b \neq a$. Add edge ab to G, to get G'. Clearly G' has an Euler circuit (just add edge ab to the Euler trail in G), and so all degrees in G' are even. This means that all degrees in G are even, except those of a and b, which are even minus one (also known as odd). In either situation, G has at most two vertices with odd degree.

  Conversely, suppose G is connected and has a most two odd degree vertices. If it has no odd degree vertices, then it has all even degree vertices and so has an Euler circuit and hence a trail. It can't have just one odd degree vertex (if it did the sum of the degrees would not be even), so he remaining case to consider is when G has exactly two odd degree vertices, a and b. Add an edge ab (or, if ab is already an edge, add a new vertex c joined only to a and b, to keep the graph simple) to get a graph which all even degrees and hence an Euler circuit. Remove ab (or ac and cb) from the circuit to get an Euler trial in the original graph, staring at a and ending at b.

- **1.4.2 5**: We'll show that if every edge lies on an odd number of cycles, then all degrees are even (so G has an Euler circuit). Consider any vertex v. The total number of cycles passing through v is *half* the sum, over all edges e incident with v, of the number of cycles using the edge e; the reason for the half is that each cycle using v uses exactly *two* of the edges incident with v, and so is counted twice in the sum. Since the number of cycles passing through v is an integer, it must be that the sum, over all edges e incident with v, of the number of cycles using the edge e, must be even. This is a sum of odd terms (by hypothesis), so for the sum to be even, there must be an even number of terms, and so the degree of v must be even, as we claimed.

- **1.4.2 7a**: To have an Euler trail, at most two vertices must have odd degree. K_{n_1,n_2} has n_1 vertices of degree n_2 and n_2 of degree n_1. We consider cases, depending on the parities of n_1 and n_2 (I'm not thinking about the possibility that $n_1, n_2 = 0$, since we didn't really define K_{n_1,n_2} in this case):

  - If n_1, n_2 are both even: all degrees are even, G has Euler circuit no matter what the values of n_1, n_2
  - If n_1 is odd, n_2 is even: G has n_2 vertices of odd degree, the rest even, so has and Euler trail (but not a circuit) only if $n_2 = 2$
  - If n_1 is even, n_2 is odd: G has n_1 vertices of odd degree, the rest even, so has and Euler trail (but not a circuit) only if $n_1 = 2$
  - If n_1, n_2 are both odd: all degrees of G are odd, and the only such G with an Euler trail is $K_{1,1}$, so must have $n_1 = n_2 = 1$

  Summary: K_{n_1,n_2} has an Euler trail iff n_1, n_2 are both even, OR exactly one of n_1, n_2 is odd and the other is 2, OR both are odd and equal to 1.

- **1.4.2 7b**: Following the analysis of the last part, K_{n_1,n_2} has an Euler circuit iff n_1, n_2 are both even.

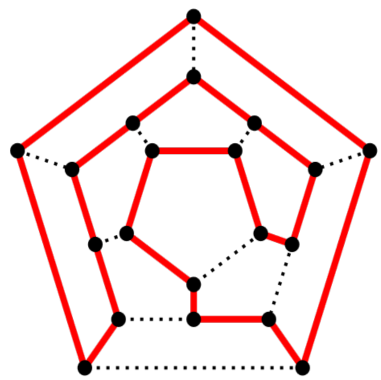
- In Problem 1.4.2(5), we showed that if every edge of G lies on an odd number of cycles, then G is Eulerian. It turns out that the converse of this is true, also (so this gives a new characterization of Eulerian graphs: a connected graph is Eulerian if and only if every vertex lies on an odd number of cycles). The proof of this is rather harder than I was expecting when I asked the question. I'll give a rough sketch here, and write up a more detailed solution later:

  Fix edge $e = uv$. We begin by counting walks from u to v in $G - uv$ that don't every immediately repeat an edge (i.e. that don't ever go $a-b-a$). Since (in $G-uv$) $d(u), d(v)$ are both odd, and all other degrees are even (this is where we use the hypothesis that G is Eulerian), when we are constructing such a walk, we always have an odd number of choices to make (Starting from u there are an odd number of choices, and from every other vertex, there is one edge we can't use (the edge we came in on) so there are an odd number of ways to continue). From this, it's possible to argue that there an odd number of such (finite) walks in total (essentially, because the product of odd numbers is odd).

  Now we go to paths from u to v. All paths appear as walks of the kind described above, but some vertex-repeating walks also appear, and we need to remove these. These vertex-repeating walks can always be deleted in pairs: given a vertex-repeating walk, find a pair of identical vertices along the walk with the property that there is no repetitions of vertices in between them. If the walk goes, for example, $a-b-c-d-e-a$ between these two repeating vertices (so the walk traverses a cycle), then there is another walk that is the same as the first except that in the middle it goes $a-e-d-c-b-a$ (so the walk traverses the cycle in the opposite direction). These two vertex-repeating walks can be removed from the list of walks together. In this way, we can eliminate all vertex-repeating walks by deleting an even number of walks; so what's left is an odd number of paths.

Since there are an odd number of paths from u to v in $G-uv$, there are an odd number of cycles using uv in G (add uv to each path to get each cycle).

- **1.4.3 1**: Here's a possible solution:



- **1.4.3 3**: Let $P = v_1 v_2 \dots v_p$ be a longest path in G (or length $p-1$). We separate two cases.

  First, suppose that $v_1 v_p$ is an edge. If so, we have a cycle $C = v_1 v_2 \dots v_p c_1$. If this includes all the vertices, we have a Hamilton cycle. If not, then since G is connected, there is a v that is not in $\{v_1, \dots, v_p\}$ that is adjacent to something in C; but then we can use the edge from v to C, together with all but one of the edges of C, to create a path of length p in G, a contradiction (P, the longest path, has length $p-1$). That deals completely with the case $v_1 v_p$ an edge.

  If $v_1 v_p$ is not an edge, then $\deg(v_1)+\deg(v_p) \geq n$, by hypothesis. We may not complete the proof exactly as we completed the proof of Dirac theorem in class; if you read through the proof, you will see that we did not actually use $\delta \geq n/2$ in that proof, only that $\deg(v_1)+\deg(v_p) \geq n$.

- **1.4.3 10a**: Let $C = v_1 v_2 \dots v_n v_1$ be a Hamilton cycle. Say that a set of vertices forms an *interval* in C if it of the form $\{v_i, v_{i+1}, \dots, v_i + k\}$ for some $i+k \leq n$; or of the form $\{v_i, v_{i+1}, \dots, v_{n-1}, v_n, v_1, v_2, \dots, v_k\}$ for some $k < i$. It's easy to see that if we remove ℓ disjoint intervals from C, then the cycle breaks into exactly ℓ components. A collection of $|S|$ vertices from G can form at most $|S|$ maximal intervals, so the removal of $|S|$ vertices breaks C into at most $|S|$ components. Adding back the edges of G that aren't in C, we can only decrease the number of components; so $G-S$ has at most $|S|$ components.

- **1.4.3 10b**: If G has a Hamilton cycle, a stronger bound follows from part a). So assume that $P = v_1 v_2 \dots v_n$ be a Hamilton path, with $v_1 \not\sim v_n$. Form G' from G by adding edge $v_1 v_n$. Removing $|S|$ vertices from G' creates at most $|S|$ components, by part a). Removing (if it's still there) the phantom edge $v_1 v_n$ adds at most one extra component; so removing $|S|$ vertices from G creates at most $|S| + 1$ components.

- **1.4.3 12a**: Suppose that n_1 and n_2 differ by more than 1. Then any path starting in the smaller partite set (say, wlog, the one of size n_1) can only visit the larger partite set at most n_1 times (before we run out of vertices in the smaller set to continue the path by), and so such a path can't be Hamiltonian. Any path starting in the larger partite set can only visit the larger partite set at most $n_1 + 1$ times (before we run out of vertices in the smaller set to continue the path by), and so such a path can't be Hamiltonian, either. So $|n_1 - n_2| \leq 1$ is necessary for a Hamilton path to exist. This condition is also sufficient: if $n_2 = n_1 + 1$, start the path in the partite set of size n_2 (in which case it ends in the same partite set, and so cannot be continued to a Hamilton cycle), and if $n_2 = n_1$, start the path anywhere (in this case, it ends in the opposite partite set, and so as long as $n_1, n_2 \geq 2$ it can be continued to a Hamilton cycle; it obviously can't if $n_1 = n_2 = 1$).

  In summary: K_{n_1,n_2} is traceable iff n_1 and n_2 differ by at most 1.

- **1.4.3 12b**: Following the reasoning of part a), a necessary and sufficient condition for K_{n_1,n_2} to be Hamiltonian is if $n_1 = n_2$ and both are greater than 1.

- **Hamilton paths/cycles in Petersen graph**: For the first part (existence of a Hamilton path), see figure 1 of the figures page.

  To see that there is no Hamilton cycle: suppose there was. Then the graph consists of a 10-cycle, and 5 other edges that are all chords of this cycle (joining two non-consecutive vertices). If all 5 of these chords join pairs of vertices that are directly opposite each other in the 10-cycle, then we have some 4-cycles (see figure 2 of the figures page); so we may assume that at least one chord joins two vertices at distance 4 along the 10-cycle; note that no chord can join vertices at distance 2 or 3 along the 10-cycle, as this would instantly give either a 3-cycle or a 4-cycle. Let that chord be e. Now look at vertex v, opposite on the 10-cycle to one of the end vertices of e. Any chord added from v creates a cycle of length 3 or 4, a contradiction (see figure 3 of the figures page). So the Petersen graph is not Hamiltonian.

  What I describe above is due to Doug West. There are other, more laborious way, to prove this fact. Here's a sketch of one: any Hamilton cycle must cross from the outer pentagon to the inner star an even number of times (otherwise, it doesn't end where it began). So there are two cases, crossing twice and crossing 4 times. In either case, after a little brute force effort, you find that you cannot end up with a cycle.

- **P versus NP 1**: We use that fact that if G is connected, then for each pair i, j of vertices (with $i \neq j$), there is a path between i and j of length at most n, so there is some $1 \leq k \leq n$ such that the ij entry of A^k is greater than 0 (where A is the adjacency matrix of G); but if G is not connected, then there is some pair i, j for which the ij entry of A^k is 0 for all $k = 1, \ldots, n$.

So, start with the n by n adjacency matrix A. Compute $A^2, A^3, \ldots, A^n$. (Time for this: $n \times n^3 = n^4$ if you use standard matrix multiplication). Now look at the ij entry of each of these n matrices, for each $i \neq j$. (Time for this $n \times n^2 = n^3$). If there is even a single i, j ($i \neq j$) for which the ij entry of each A^k is 0, then G is not connected; if for every i, j ($i \neq j$) there is some k for which the ij entry of some A^k is 0, then G is connected.

The total time for this process is around n^4.

- **P versus NP 2**: For each of the no more than n^2 edges, compute the adjacency matrix of G minus that edge (this just requires changing two entries of the adjacency matrix of G). Then use the method described in the last question to check if the modified graph is connected (if it is, the edge is not a bridge; if it not, the edge is a bridge). The total time to check every edge is around $n^2 \times n^4 = n^6$.

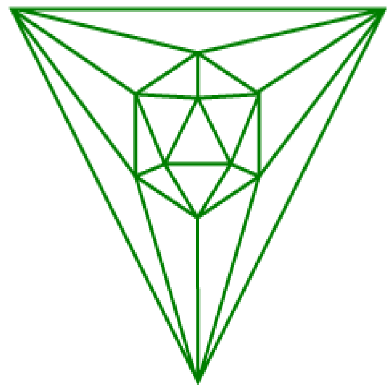
- **1.5.1 1**: See figure 4 of the figures page for possible representations.

- **1.5.1 4**: Here we have to be careful. It's not enough to say, for example, "remove an edge attached to a leaf; draw the smaller tree in the plane without crossing edges (ok by induction), then put in the last edge". If you take this approach, you need to give a convincing argument that the edges of the smaller tree have been drawn in such a way that a new edge can be added without crossing one of the previous edges, and this brings in a lot of messy topology.

  It's helpful actually to prove something a little stronger: every tree has a planar representation in which all edges are straight lines. We proceed by induction on n, the number of vertices. If $n = 1$, it is obvious. If $n > 1$, find a leaf u with unique neighbor v, and delete u to get a smaller tree, which by induction has a straight-line planar representation. Look at vertex v. In some small disc in $\mathbb{R}^2$ centered at v, the planar representation consists of a finite number of radii centered at v, dividing the disc into finitely many wedges. Pick one of these arbitrarily, put u in it (anywhere) and join it to v, giving a straight-line planar representation of the larger tree. This ends the induction step.

  Note: It's a theorem (proved independently by Fáry and Wagner) that *every* planar graph has a planar representation in which every edge is a straight line.

- **1.5.1 5**: Here's a possible solution:



- **1.5.1 8**: We know that for connected graphs, the numbers of vertices and edges determine the number of regions, so it seems very unlikely that this information determines (planar) graphs up isomorphism. Figure 5 of the figures page shows one of (many) pairs that give a counterexample.