Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 1 — Solutions

- **1.1.1 1**: See figure 1 of the figures page for one possible representation.

- **1.1.1 4**: Represent the picture as graph with four vertices (one for each of the four land masses) and seven edges (one for each of the bridges; an edge joins two vertices if there is a bridge joining the corresponding land masses). The graph (actually a multi-graph) is shown in figure 2 of the figures page.

  Notice that each vertex has odd degree (each land mass has an odd number of bridges leaving it). Suppose there was a walk that crossed each edge once and only once. Let's say it starts at vertex A and ends at vertex D. Think about what happens at vertex C. At some point you enter it for the first time, using one of its edges. Then you leave it immediately, using another edge. Since you've used two edges, and C has odd degree, it still has an odd number of edges left to be used after first visit is finished. But because C is neither the starting nor the ending vertex, every visit to it will use exactly two edges, there will always be an odd number of edges left to be used, and you will never get down to zero edges left (zero is even). The same argument works no matter what your choice of starting and ending vertex (whether they are the same vertex or different from one another).

  The problem is that the graph has too many odd degree vertices; we will return to this later and make a comprehensive theory.

- **1.1.2 1**: The maximum is achieved when each vertex has the maximum possible degree, so $n-1$. So the maximum possible sum of the degrees is $n(n-1)$ (and this can be reached, by the complete graph on n vertices). Since the sum of the degrees is twice the number of edges, the maximum number of edges is $n(n-1)/2$ (also known as $\binom{n}{2}$).

- **1.1.2 2**: If G has order $n \geq 2$, the possible degrees for vertices are $0, 1, 2, \ldots, n-1$ (a total of n possibilities). So the **only** way that a graph of order $n \geq 2$ can have n distinct degrees is if the degrees are exactly $0, 1, \ldots, n-1$. But it's impossible for G to have a vertex u of degree 0 (adjacent to nothing) and simultaneously a vertex v of degree $n-1$ (adjacent to everything) – the condition on u says that uv is not an edge, the condition on v says that it is.

  Note that if $n = 1$ then 0 and $n-1$ are equal, and it's possible for $u = v$ (indeed, the stump graph K_1 exhibits this behavior). This graph shows that the statement is not true for $n = 1$.

- **1.1.2 5**:
  - **Part a**: Suppose that the longest path is of length $j < k$. Let $v_1, v_2, \ldots, v_{j+1}$ be the vertices of a longest path. Since v_{j+1} has at least k neighbors, it must have some neighbors other than $v_1, \ldots, v_j$. Any of these neighbors can be used to extend the path, contradicting the fact that it is a longest path. So the longest path has length at least k.
  - **Part b**: Start with a longest path, say $v_1, v_2, \ldots, v_m$. Because it's a longest path, all the neighbors of v_1 must lie along the path. Because v_1 has at least k neighbors, it must be adjacent to some v_j with $j \geq k+1$. The vertices $v_1, v_2, \ldots, v_j$ are then the vertices of a cycle of length at least $k+1$.
- **1.1.2 6**: We prove this statement by induction on n, the length of the closed odd-length walk. The smallest odd value of n for which there is a closed walk of length n is $n = 3$. All such walks are necessarily cycles (there's not enough room in three edges to repeat an edge and still close the walk), so we take this as our base case.

  Fix $n \geq 5$ odd, and a length n closed walk, starting and ending at u. If the walk does not repeat a vertex (except the starting and ending u), then it is already a cycle, and we are done. So the walk does repeat a vertex. If it repeats a vertex $w \neq u$, then we can view the walk as the first "figure-of-8" pattern shown on figure 3 of the figures page (in which there may be repeated edges and/or vertices within each loop; that's ok). If the walk repeats u somewhere in the middle, we can view the walk as the second "figure-of-8" pattern shown on figure 3 of the figures page.

  In each case, since the total length of each figure-of-8 is odd, one of the loops must have odd length (if both loops in each case were even, the total length would be even). This odd loop is then an odd-length closed walk of length shorter than n. By induction, this walk has an odd cycle, which is also an odd cycle contained in the longer walk.
- **1.1.2 10**: Imagine building the graph up from the empty graph, by adding one vertex at a time. Initially the graph has n components. The addition of each edge either joins two vertices in different components, and so reduces the number of components by one, or joins two vertices already in the same component, so leaves the number of components the same. After adding $n-2$ edges, the lowest possible number of components remaining is therefore $n-(n-2) = 2$; so $n-2$ edges (or fewer) is not enough to drop the number of components to 1 (i.e., make the graph connected).
- **1.1.2 16**:
  - **Part a**: Not only is G connected, it has diameter at most 2. To see this, take vertices $u \neq v$. If uv is an edge, everything is ok. If not, u has at least $(n-1)/2$ neighbors, none of which is u or v, and v has at least $(n-1)/2$ neighbors, none of which is u or v. Since $(n-1)/2 + (n-1)/2 = n-1 > n-2$, and $n-2$ is the number of vertices other than u and v, these two sets of neighbors must have a vertex in common, w say, and u-w-v is a u-v path of length 2.

- **Part b**: Let n be even, and let G be the graph consisting of two disjoint copies of $K_{n/2}$, the complete graph on $n/2$ vertices (see figure 4 of the figures page). This graph has n vertices, it is $(n/2)-1=(n-2)/2$-regular and so has minimum degree $(n-2)/2$, and is not connected.

- **1.1.3 3**: K_4 has a 3-cycle (C_3) as a subgraph, so if K_4 was a subgraph of $K_{4,4}$, then we would also have C_3 as a subgraph of $K_{4,4}$. But $K_{4,4}$ is, by definition, bipartite, and bipartite graphs cannot have odd cycles as subgraphs. So K_4 is **not** a subgraph of $K_{4,4}$.

- **1.1.3 8**: Let $f: V(G)\to V(H)$ be an isomorphism from G to H. I claim that the same f is an isomorphism from $\overline{G}$ to $\overline{H}$ (note that $V(G)=V(\overline{G})$ and $V(H)=V(\overline{H})$, so f is a bijection between $V(\overline{G}$ and $V(\overline{H})$.

  To see that f works, note that

$$\begin{aligned} uv\in E(\overline{G}) &\iff uv\notin E(G)\\ &\iff f(u)f(v)\notin E(H)\\ &\iff f(u)f(v)\in E(\overline{H}). \end{aligned}$$

  In the middle line, I used the fact that f is an isomorphism from G to H.

- **1.1.3 10**:
  - **Part a**: P and Q are isomorphic. With the vertices labeled as in figure 5 on the figures page, a valid isomorphism is $f(1)=a$, $f(2)=b$, $f(3)=c$, $\ldots$, $f(10)=j$ (and there are many other possible answers).

    Here's one way to come up with this isomorphism: start with a "partial isomorphism", say by sending 5-cycle 1-2-3-4-5 to 5-cycle a-b-c-d-e, and seeing if it extends to a full isomorphism. In this case it does, in a forced way. For example, 1 is adjacent to 5, 2 and 6, and a is adjacent to e, b and f. Since the partial isomorphism has already dealt with the edges 15 and 12 by sending 1 to a, 5 to e and 2 to b, it **must** deal with the edge 16 by sending 6 to f, the last remaining unaccounted for neighbor of a in Q. Similarly all other choices are forced, and the result turns out to be an isomorphism. (It doesn't always work out like this. Partial isomorphisms don't always extend to full isomorphisms. But the Petersen graph (P and Q) is so symmetric, it turns out that any partial isomorphism that begins by sending a 5-cycle to a 5-cycle extends to a unique isomorphism.)
  - **Part b**: Q and R are not isomorphic; R has a 4-cycle, and Q does not. If Q and R were isomorphic, then an isomorphism would move the 4-cycle in R to a 4-cycle in Q.

- **1.2.1 2**:
  - P_{2k}: radius k, diameter $2k-1$
  - P_{2k+1}: radius $k+1$, diameter $2k$
  - C_{2k}: radius k, diameter k

- C_{2k+1}: radius k, diameter k
- K_n: radius 1, diameter 1 (unless $n = 1$, in which case radius 0, diameter 0)
- $K_{n,m}$: Assume without loss of generality that $n \leq m$. If $n = 1$ and $m = 1$ then radius 1, diameter 1; if $n = 1$ and $m > 1$ then radius 1, diameter 1; if $n > 1$ then radius 2, diameter 2

- **1.2.1 3**: P_{2k} has two vertices in the center, and P_{2k+1} has one (see figure 6 on the figures page).

- **1.2.1 7**:
  - **Part a**: See figure 7 of the figures page
  - **Part b**: Two vertices are connected in $G^{\text{diam}(G)}$ if their distance is at most diamG; but by definition of diameter, all pairs of vertices are at most this distance apart. So all pairs are connected, and $G^{\text{diam}(G)}$ is K_n (the complete graph on n vertices).

- **1.2.1 8(d)**: One way to obtain radius r and diameter d, where $r \leq d \leq 2r$ are arbitrary, is to start with a cycle of length $2r$, and add a path of length $d - r$ from one of the vertices of the cycle. See the figure 8 on the figures page for two examples: first $r = 3$ and $d = 5$ and second $r = 4$ and $d = 8$. The vertices are labeled with their eccentricities (distance to the furthest vertex). Note that the vertex where the cycle meets the path is labeled r in each case, the vertex at the far end of the path is labeled d, and these are the extreme labels. This phenomenon holds for general r and d.

- **1.2.1 11(a)**: Suppose that $\overline{G}$ is *not* connected. Let X be the vertex set of a component of $\overline{G}$, and let Y be the rest of the vertices. Because there are no vertices from X to Y in $\overline{G}$, it follows that in G, every vertex of X is adjacent to every vertex of Y. That means that there a path in G between any two vertices with length at most 2, contradicting the fact that $\text{diam}(G) \geq 3$. So $\overline{G}$ must be connected.

- **Extra problem 1**: The statement turns out to be very false. Probably the simplest counterexample is on the graph K_2 (a single edge), say with vertices a and b. The closed walk a-b-a has length 2, but clearly does not contain an even cycle!

- **Extra problem 2**: If G is connected, there's basically only one way to create a bipartition: pick a vertex v, put it in the first of the partite sets, then build the bipartition by putting the neighbours of v in the second of the partite sets, then putting the neighbours of the neighbours in the first, etc. The only leeway this gives us is in the order in which we name the partite sets. So for a connected graph, there are exactly two bipartitions.

  If G has a number, say k, of components, then we get a bipartition of G be bipartitioning each component into $X_i \cup Y_i$, $i = 1, \ldots k$, and then forming a bipartition of G by taking $X = \cup_i X_i$ and $Y = \cup Y_i$. Since there are two choices for (X_i, Y_i) for each i, this leads to 2^k choices for a "grand" bipartition.

  Conclusion: Bipartite G has $2^{c(G)}$ bipartitions, where $c(G)$ is the number of components of G.