Examples involving conditional probability

Math 30530, Fall 2013

September 5, 2013

I'm **always** late to work

60% of days I walk to work. On those days

$\longrightarrow$ I'm late 80% of the time

$\longrightarrow$ I'm on time 20% of the time

40% of days I drive to work. On those days

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Answer: $\Pr(L) = \Pr(L \cap W) + \Pr(L \cap D) =$
$\Pr(W)\Pr(L|W) + \Pr(D)\Pr(L|D) = (.6)(.8) + (.4)(.5) = .68$

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Question 2: On a day that I'm late for work, what's the probability that I drove?

Answer: $\Pr(D|L) = \frac{\Pr(D \cap L)}{\Pr(L)} = \frac{(.4)(.5)}{.68} \approx .29.$

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I draw 5 cards from a well-shuffled deck.

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$$\begin{aligned}
\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) &= \Pr(A_1) \times \Pr(A_2|A_1) \times \\
&\quad \Pr(A_3|A_1 \cap A_2) \times \Pr(A_4|A_1 \cap A_2 \cap A_3) \times \\
&\quad \Pr(A_5|A_1 \cap A_2 \cap A_3 \cap A_4) \\
&= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right)
\end{aligned}$$

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&\quad \Pr(A_5|A_1 \cap A_2 \cap A_3 \cap A_4) \\
&= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right) \\
&= \frac{13 \times 12 \times 11 \times 10 \times 9}{52 \times 51 \times 50 \times 49 \times 48}
\end{aligned}$$

Am I at the gym?

- If I go to the gym today, there's an 80% chance that I'll go tomorrow
- If I skip the gym today, there's a 40% chance that I'll go tomorrow
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$$\begin{aligned} \Pr(G_0) &= 1 \\ \Pr(G_1) &= .8 \end{aligned}$$

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\Pr(G_2) &= \Pr(G_2 \cap G_1) + \Pr(G_2 \cap G_1^c) \\
&= \Pr(G_1)\Pr(G_2|G_1) + \Pr(G_1^c)\Pr(G_2|G_1^c) \\
&= .8\Pr(G_1) + .4(1 - \Pr(G_1)) \\
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In general,

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$\Pr(G_{30}) \approx \frac{2}{3} + 10^{-12}$, and for n above about 10, $\Pr(G_n)$ basically indistinguishable from $2/3$