CHAPTER 4

Solution to Problem 4.1. Let $Y = \sqrt{|X|}$. We have, for $0 \le y \le 1$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\sqrt{|X|} \le y) = \mathbf{P}(-y^2 \le X \le y^2) = y^2,$$

and therefore by differentiation,

$$f_Y(y) = 2y, \qquad \text{for } 0 \le y \le 1.$$

Let $Y = -\ln |X|$. We have, for $y \ge 0$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\ln |X| \ge -y) = \mathbf{P}(X \ge e^{-y}) + \mathbf{P}(X \le -e^{-y}) = 1 - e^{-y},$$

and therefore by differentiation

$$f_Y(y) = e^{-y}, \qquad \text{for } y \ge 0,$$

so Y is an exponential random variable with parameter 1. This exercise provides a method for simulating an exponential random variable using a sample of a uniform random variable.

Solution to Problem 4.2. Let $Y = e^X$. We first find the CDF of Y, and then take the derivative to find its PDF. We have

$$\mathbf{P}(Y \le y) = \mathbf{P}(e^X \le y) = \begin{cases} \mathbf{P}(X \le \ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_Y(y) = \begin{cases} \dfrac{d}{dx} F_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \dfrac{1}{y} f_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

When X is uniform on $[0, 1]$, the answer simplifies to

$$f_Y(y) = \begin{cases} \dfrac{1}{y}, & \text{if } 0 < y \le e, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.3. Let $Y = |X|^{1/3}$. We have

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\big(|X|^{1/3} \le y\big) = \mathbf{P}\big(-y^3 \le X \le y^3\big) = F_X(y^3) - F_X(-y^3),$$

and therefore, by differentiating,

$$f_Y(y) = 3y^2 f_X(y^3) + 3y^2 f_X(-y^3), \qquad \text{for } y > 0.$$

Let $Y = |X|^{1/4}$. We have

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}\big(|X|^{1/4} \leq y\big) = \mathbf{P}(-y^4 \leq X \leq y^4) = F_X(y^4) - F_X(-y^4),$$

and therefore, by differentiating,

$$f_Y(y) = 4y^3 f_X(y^4) + 4y^3 f_X(-y^4), \qquad \text{for } y > 0.$$

Solution to Problem 4.4. We have

$$F_Y(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ \mathbf{P}(5 - y \leq X \leq 5) + \mathbf{P}(20 - y \leq X \leq 20), & \text{if } 0 \leq y \leq 5, \\ \mathbf{P}(20 - y \leq X \leq 20), & \text{if } 5 < y \leq 15, \\ 1, & \text{if } y > 15. \end{cases}$$

Using the CDF of X, we have

$$\mathbf{P}(5 - y \leq X \leq 5) = F_X(5) - F_X(5 - y),$$

$$\mathbf{P}(20 - y \leq X \leq 20) = F_X(20) - F_X(20 - y).$$

Thus,

$$F_Y(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ F_X(5) - F_X(5 - y) + F_X(20) - F_X(20 - y), & \text{if } 0 \leq y \leq 5, \\ F_X(20) - F_X(20 - y), & \text{if } 5 < y \leq 15, \\ 1, & \text{if } y > 15. \end{cases}$$

Differentiating, we obtain

$$f_Y(y) = \begin{cases} f_X(5 - y) + f_X(20 - y), & \text{if } 0 \leq y \leq 5, \\ f_X(20 - y), & \text{if } 5 < y \leq 15, \\ 0, & \text{otherwise,} \end{cases}$$

consistent with the result of Example 3.14.

Solution to Problem 4.5. Let $Z = |X - Y|$. We have

$$F_Z(z) = P\big(|X - Y| \leq z\big) = 1 - (1 - z)^2.$$

(To see this, draw the event of interest as a subset of the unit square and calculate its area.) Taking derivatives, the desired PDF is

$$f_Z(z) = \begin{cases} 2(1 - z), & \text{if } 0 \leq z \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.6. Let $Z = |X - Y|$. To find the CDF, we integrate the joint PDF of X and Y over the region where $|X - Y| \leq z$ for a given z. In the case where $z \leq 0$ or $z \geq 1$, the CDF is 0 and 1, respectively. In the case where $0 < z < 1$, we have

$$F_Z(z) = \mathbf{P}(X - Y \leq z,\, X \geq Y) + \mathbf{P}(Y - X \leq z,\, X < Y).$$

The events $\{X - Y \leq z,\, X \geq Y\}$ and $\{Y - X \leq z,\, X < Y\}$ can be identified with subsets of the given triangle. After some calculation using triangle geometry, the areas of these subsets can be verified to be $z/2 + z^2/4$ and $1/4 - (1-z)^2/4$, respectively. Therefore, since $f_{X,Y}(x,y) = 1$ for all (x,y) in the given triangle,

$$F_Z(z) = \left(\frac{z}{2} + \frac{z^2}{4}\right) + \left(\frac{1}{4} - \frac{(1-z)^2}{4}\right) = z.$$

Thus,

$$F_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ z, & \text{if } 0 < z < 1, \\ 1, & \text{if } z \geq 1. \end{cases}$$

By taking the derivative with respect to z, we obtain

$$f_Z(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.7. Let X and Y be the two points, and let $Z = \max\{X, Y\}$. For any $t \in [0, 1]$, we have

$$\mathbf{P}(Z \leq t) = \mathbf{P}(X \leq t)\mathbf{P}(Y \leq t) = t^2,$$

and by differentiating, the corresponding PDF is

$$f_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ 2z, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z \geq 1. \end{cases}$$

Thus, we have

$$\mathbf{E}[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_0^1 2z^2 dz = \frac{2}{3}.$$

The distance of the largest of the two points to the right endpoint is $1 - Z$, and its expected value is $1 - \mathbf{E}[Z] = 1/3$. A symmetric argument shows that the distance of the smallest of the two points to the left endpoint is also 1/3. Therefore, the expected distance between the two points must also be 1/3.

Solution to Problem 4.8. Note that $f_X(x)$ and $f_Y(z - x)$ are nonzero only when $x \geq 0$ and $x \leq z$, respectively. Thus, in the convolution formula, we only need to integrate for x ranging from 0 to z:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx = \lambda^2 e^{-z} \int_0^z dx = \lambda^2 z e^{-\lambda z}.$$

Solution to Problem 4.9. Let $Z = X - Y$. We will first calculate the CDF $F_Z(z)$ by considering separately the cases $z \geq 0$ and $z < 0$. For $z \geq 0$, we have (see the left side of Fig. 4.6)

$$\begin{aligned}
F_Z(z) &= \mathbf{P}(X - Y \leq z) \\
&= 1 - \mathbf{P}(X - Y > z) \\
&= 1 - \int_0^\infty \left(\int_{z+y}^\infty f_{X,Y}(x, y)\, dx \right) dy \\
&= 1 - \int_0^\infty \mu e^{-\mu y} \left(\int_{z+y}^\infty \lambda e^{-\lambda x}\, dx \right) dy \\
&= 1 - \int_0^\infty \mu e^{-\mu y} e^{-\lambda(z+y)}\, dy \\
&= 1 - e^{-\lambda z} \int_0^\infty \mu e^{-(\lambda+\mu)y}\, dy \\
&= 1 - \frac{\mu}{\lambda + \mu} e^{-\lambda z}.
\end{aligned}$$

For the case $z < 0$, we have using the preceding calculation

$$F_Z(z) = 1 - F_Z(-z) = 1 - \left(1 - \frac{\lambda}{\lambda + \mu} e^{-\mu(-z)}\right) = \frac{\lambda}{\lambda + \mu} e^{\mu z}.$$

Combining the two cases $z \geq 0$ and $z < 0$, we obtain

$$F_Z(z) = \begin{cases} 1 - \dfrac{\mu}{\lambda + \mu} e^{-\lambda z}, & \text{if } z \geq 0, \\ \dfrac{\lambda}{\lambda + \mu} e^{\mu z}, & \text{if } z < 0. \end{cases}$$

The PDF of Z is obtained by differentiating its CDF. We have

$$f_Z(z) = \begin{cases} \dfrac{\lambda\mu}{\lambda + \mu} e^{-\lambda z}, & \text{if } z \geq 0, \\ \dfrac{\lambda\mu}{\lambda + \mu} e^{\mu z}, & \text{if } z < 0. \end{cases}$$

For an alternative solution, fix some $z \geq 0$ and note that $f_Y(x - z)$ is nonzero only when $x \geq z$. Thus,

$$\begin{aligned}
f_{X-Y}(z) &= \int_{-\infty}^\infty f_X(x) f_Y(x - z)\, dx \\
&= \int_z^\infty \lambda e^{-\lambda x} \mu e^{-\mu(x-z)}\, dx \\
&= \lambda\mu e^{\lambda z} \int_z^\infty e^{-(\lambda+\mu)x}\, dx \\
&= \lambda\mu e^{\lambda z} \frac{1}{\lambda + \mu} e^{-(\lambda+\mu)z} \\
&= \frac{\lambda\mu}{\lambda + \mu} e^{-\mu z},
\end{aligned}$$