$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 1 - e^{-\lambda z}, & \text{if } z \geq 0. \end{cases}$$

We have $f_X(x) = pf_Y(x) + (1-p)f_Z(x)$, and consequently $F_X(x) = pF_Y(x) + (1-p)F_Z(x)$. It follows that

$$F_X(x) = \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ p + (1-p)(1 - e^{-\lambda x}), & \text{if } x \geq 0, \end{cases}$$
$$= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ 1 - (1-p)e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

Solution to Problem 3.11. (a) X is a standard normal, so by using the normal table, we have $\mathbf{P}(X \leq 1.5) = \Phi(1.5) = 0.9332$. Also $\mathbf{P}(X \leq -1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

(b) The random variable $(Y-1)/2$ is obtained by subtracting from Y its mean (which is 1) and dividing by the standard deviation (which is 2), so the PDF of $(Y-1)/2$ is the standard normal.

(c) We have, using the normal table,

$$\begin{aligned} \mathbf{P}(-1 \leq Y \leq 1) &= \mathbf{P}\big(-1 \leq (Y-1)/2 \leq 0\big) \\ &= \mathbf{P}(-1 \leq Z \leq 0) \\ &= \mathbf{P}(0 \leq Z \leq 1) \\ &= \Phi(1) - \Phi(0) \\ &= 0.8413 - 0.5 \\ &= 0.3413, \end{aligned}$$

where Z is a standard normal random variable.

Solution to Problem 3.12. The random variable $Z = X/\sigma$ is a standard normal, so

$$\mathbf{P}(X \geq k\sigma) = \mathbf{P}(Z \geq k) = 1 - \Phi(k).$$

From the normal tables we have

$$\Phi(1) = 0.8413, \qquad \Phi(2) = 0.9772, \qquad \Phi(3) = 0.9986.$$

Thus $\mathbf{P}(X \geq \sigma) = 0.1587$, $\mathbf{P}(X \geq 2\sigma) = 0.0228$, $\mathbf{P}(X \geq 3\sigma) = 0.0014$.

We also have

$$\mathbf{P}\big(|X| \leq k\sigma\big) = \mathbf{P}\big(|Z| \leq k\big) = \Phi(k) - \mathbf{P}(Z \leq -k) = \Phi(k) - \big(1 - \Phi(k)\big) = 2\Phi(k) - 1.$$

Using the normal table values above, we obtain

$$\mathbf{P}(|X| \leq \sigma) = 0.6826, \qquad \mathbf{P}(|X| \leq 2\sigma) = 0.9544, \qquad \mathbf{P}(|X| \leq 3\sigma) = 0.9972,$$

where t is a standard normal random variable.

Solution to Problem 3.13. Let X and Y be the temperature in Celsius and Fahrenheit, respectively, which are related by $X = 5(Y - 32)/9$. Therefore, 59 degrees Fahrenheit correspond to 15 degrees Celsius. So, if Z is a standard normal random variable, we have using $\mathbf{E}[X] = \sigma_X = 10$,

$$\mathbf{P}(Y \le 59) = \mathbf{P}(X \le 15) = \mathbf{P}\left(Z \le \frac{15 - \mathbf{E}[X]}{\sigma_X}\right) = \mathbf{P}(Z \le 0.5) = \Phi(0.5).$$

From the normal tables we have $\Phi(0.5) = 0.6915$, so $\mathbf{P}(Y \le 59) = 0.6915$.

Solution to Problem 3.15. (a) Since the area of the semicircle is $\pi r^2/2$, the joint PDF of X and Y is $f_{X,Y}(x, y) = 2/\pi r^2$, for (x, y) in the semicircle, and $f_{X,Y}(x, y) = 0$, otherwise.

(b) To find the marginal PDF of Y, we integrate the joint PDF over the range of X. For any possible value y of Y, the range of possible values of X is the interval $[-\sqrt{r^2 - y^2}, \sqrt{r^2 - y^2}]$, and we have

$$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\, dx = \begin{cases} \dfrac{4\sqrt{r^2 - y^2}}{\pi r^2}, & \text{if } 0 \le y \le r, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{E}[Y] = \frac{4}{\pi r^2} \int_0^r y\sqrt{r^2 - y^2}\, dy = \frac{4r}{3\pi},$$

where the integration is performed using the substitution $z = r^2 - y^2$.

(c) There is no need to find the marginal PDF f_Y in order to find $\mathbf{E}[Y]$. Let D denote the semicircle. We have, using polar coordinates

$$\mathbf{E}[Y] = \int\int_{(x,y)\in D} y f_{X,Y}(x, y)\, dx\, dy = \int_0^\pi \int_0^r \frac{2}{\pi r^2}\, s(\sin\theta) s\, ds\, d\theta = \frac{4r}{3\pi}.$$

Solution to Problem 3.16. Let A be the event that the needle will cross a horizontal line, and let B be the probability that it will cross a vertical line. From the analysis of Example 3.11, we have that

$$\mathbf{P}(A) = \frac{2l}{\pi a}, \qquad \mathbf{P}(B) = \frac{2l}{\pi b}.$$

Since at most one horizontal (or vertical) line can be crossed, the expected number of horizontal lines crossed is $\mathbf{P}(A)$ [or $\mathbf{P}(B)$, respectively]. Thus the expected number of crossed lines is

$$\mathbf{P}(A) + \mathbf{P}(B) = \frac{2l}{\pi a} + \frac{2l}{\pi b} = \frac{2l(a + b)}{\pi ab}.$$

The probability that at least one line will be crossed is

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$