CHAPTER 3

Solution to Problem 3.1. The random variable $Y = g(X)$ is discrete and its PMF is given by

$$p_Y(1) = \mathbf{P}(X \le 1/3) = 1/3, \qquad p_Y(2) = 1 - p_Y(1) = 2/3.$$

Thus,

$$\mathbf{E}[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.$$

The same result is obtained using the expected value rule:

$$\mathbf{E}[Y] = \int_0^1 g(x) f_X(x)\, dx = \int_0^{1/3} dx + \int_{1/3}^1 2\, dx = \frac{5}{3}.$$

Solution to Problem 3.2. We have

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda |x|}\, dx = 2 \cdot \frac{1}{2} \int_0^{\infty} \lambda e^{-\lambda x}\, dx = 2 \cdot \frac{1}{2} = 1,$$

where we have used the fact $\int_0^\infty \lambda e^{-\lambda x} dx = 1$, i.e., the normalization property of the exponential PDF. By symmetry of the PDF, we have $\mathbf{E}[X] = 0$. We also have

$$\mathbf{E}[X^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda |x|} dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

where we have used the fact that the second moment of the exponential PDF is $2/\lambda^2$. Thus

$$\mathrm{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = 2/\lambda^2.$$

Solution to Problem 3.5. Let $A = bh/2$ be the area of the given triangle, where b is the length of the base, and h is the height of the triangle. From the randomly chosen point, draw a line parallel to the base, and let A_x be the area of the triangle thus formed. The height of this triangle is $h - x$ and its base has length $b(h - x)/h$. Thus $A_x = b(h - x)^2/(2h)$. For $x \in [0, h]$, we have

$$F_X(x) = 1 - \mathbf{P}(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h-x)^2/(2h)}{bh/2} = 1 - \left(\frac{h-x}{h}\right)^2,$$

while $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > h$.

The PDF is obtained by differentiating the CDF. We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \dfrac{2(h-x)}{h^2}, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 3.6. Let X be the waiting time and Y be the number of customers found. For $x < 0$, we have $F_X(x) = 0$, while for $x \geq 0$,

$$F_X(x) = \mathbf{P}(X \leq x) = \frac{1}{2}\mathbf{P}(X \leq x \,|\, Y = 0) + \frac{1}{2}\mathbf{P}(X \leq x \,|\, Y = 1).$$

Since

$$\mathbf{P}(X \leq x \,|\, Y = 0) = 1,$$

$$\mathbf{P}(X \leq x \,|\, Y = 1) = 1 - e^{-\lambda x},$$

we obtain

$$F_X(x) = \begin{cases} \frac{1}{2}(2 - e^{-\lambda x}), & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the CDF has a discontinuity at $x = 0$. The random variable X is neither discrete nor continuous.

Solution to Problem 3.7. (a) We first calculate the CDF of X. For $x \in [0, r]$, we have

$$F_X(x) = \mathbf{P}(X \leq x) = \frac{\pi x^2}{\pi r^2} = \left(\frac{x}{r}\right)^2.$$

For $x < 0$, we have $F_X(x) = 0$, and for $x > r$, we have $F_X(x) = 1$. By differentiating, we obtain the PDF

$$f_X(x) = \begin{cases} \frac{2x}{r^2}, & \text{if } 0 \leq x \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\mathbf{E}[X] = \int_0^r \frac{2x^2}{r^2}\,dx = \frac{2r}{3}.$$

Also

$$\mathbf{E}[X^2] = \int_0^r \frac{2x^3}{r^2}\,dx = \frac{r^2}{2},$$

so

$$\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}.$$

(b) Alvin gets a positive score in the range $[1/t, \infty)$ if and only if $X \leq t$, and otherwise he gets a score of 0. Thus, for $s < 0$, the CDF of S is $F_S(s) = 0$. For $0 \leq s < 1/t$, we have

$$F_S(s) = \mathbf{P}(S \leq s) = \mathbf{P}(\text{Alvin's hit is outside the inner circle}) = 1 - \mathbf{P}(X \leq t) = 1 - \frac{t^2}{r^2}.$$

For $1/t < s$, the CDF of S is given by

$$F_S(s) = \mathbf{P}(S \leq s) = \mathbf{P}(X \leq t)\mathbf{P}(S \leq s \,|\, X \leq t) + \mathbf{P}(X > t)\mathbf{P}(S \leq s \,|\, X > t).$$

We have

$$\mathbf{P}(X \le t) = \frac{t^2}{r^2}, \qquad \mathbf{P}(X > t) = 1 - \frac{t^2}{r^2},$$

and since $S = 0$ when $X > t$,

$$\mathbf{P}(S \le s \,|\, X > t) = 1.$$

Furthermore,

$$\mathbf{P}(S \le s \,|\, X \le t) = \mathbf{P}(1/X \le s \,|\, X \le t) = \frac{\mathbf{P}(1/s \le X \le t)}{\mathbf{P}(X \le t)} = \frac{\dfrac{\pi t^2 - \pi(1/s)^2}{\pi r^2}}{\dfrac{\pi t^2}{\pi r^2}} = 1 - \frac{1}{s^2 t^2}.$$

Combining the above equations, we obtain

$$\mathbf{P}(S \le s) = \frac{t^2}{r^2}\left(1 - \frac{1}{s^2 t^2}\right) + 1 - \frac{t^2}{r^2} = 1 - \frac{1}{s^2 r^2}.$$

Collecting the results of the preceding calculations, the CDF of S is

$$F_S(s) = \begin{cases} 0, & \text{if } s < 0, \\ 1 - \dfrac{t^2}{r^2}, & \text{if } 0 \le s < 1/t, \\ 1 - \dfrac{1}{s^2 r^2}, & \text{if } 1/t \le s. \end{cases}$$

Because F_S has a discontinuity at $s = 0$, the random variable S is not continuous.

Solution to Problem 3.8. (a) By the total probability theorem, we have

$$F_X(x) = \mathbf{P}(X \le x) = p\mathbf{P}(Y \le x) + (1-p)\mathbf{P}(Z \le x) = pF_Y(x) + (1-p)F_Z(x).$$

By differentiating, we obtain

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Consider the random variable Y that has PDF

$$f_Y(y) = \begin{cases} \lambda e^{\lambda y}, & \text{if } y < 0 \\ 0, & \text{otherwise,} \end{cases}$$

and the random variable Z that has PDF

$$f_Z(z) = \begin{cases} \lambda e^{-\lambda z}, & \text{if } y \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

We note that the random variables $-Y$ and Z are exponential. Using the CDF of the exponential random variable, we see that the CDFs of Y and Z are given by

$$F_Y(y) = \begin{cases} e^{\lambda y}, & \text{if } y < 0, \\ 1, & \text{if } y \ge 0, \end{cases}$$

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 1 - e^{-\lambda z}, & \text{if } z \geq 0. \end{cases}$$

We have $f_X(x) = pf_Y(x) + (1-p)f_Z(x)$, and consequently $F_X(x) = pF_Y(x) + (1-p)F_Z(x)$. It follows that

$$\begin{aligned} F_X(x) &= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ p + (1-p)(1 - e^{-\lambda x}), & \text{if } x \geq 0, \end{cases} \\ &= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ 1 - (1-p)e^{-\lambda x}, & \text{if } x \geq 0. \end{cases} \end{aligned}$$

Solution to Problem 3.11. (a) X is a standard normal, so by using the normal table, we have $\mathbf{P}(X \leq 1.5) = \Phi(1.5) = 0.9332$. Also $\mathbf{P}(X \leq -1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

(b) The random variable $(Y-1)/2$ is obtained by subtracting from Y its mean (which is 1) and dividing by the standard deviation (which is 2), so the PDF of $(Y-1)/2$ is the standard normal.

(c) We have, using the normal table,

$$\begin{aligned} \mathbf{P}(-1 \leq Y \leq 1) &= \mathbf{P}\big(-1 \leq (Y-1)/2 \leq 0\big) \\ &= \mathbf{P}(-1 \leq Z \leq 0) \\ &= \mathbf{P}(0 \leq Z \leq 1) \\ &= \Phi(1) - \Phi(0) \\ &= 0.8413 - 0.5 \\ &= 0.3413, \end{aligned}$$

where Z is a standard normal random variable.

Solution to Problem 3.12. The random variable $Z = X/\sigma$ is a standard normal, so

$$\mathbf{P}(X \geq k\sigma) = \mathbf{P}(Z \geq k) = 1 - \Phi(k).$$

From the normal tables we have

$$\Phi(1) = 0.8413, \qquad \Phi(2) = 0.9772, \qquad \Phi(3) = 0.9986.$$

Thus $\mathbf{P}(X \geq \sigma) = 0.1587$, $\mathbf{P}(X \geq 2\sigma) = 0.0228$, $\mathbf{P}(X \geq 3\sigma) = 0.0014$.

We also have

$$\mathbf{P}\big(|X| \leq k\sigma\big) = \mathbf{P}\big(|Z| \leq k\big) = \Phi(k) - \mathbf{P}(Z \leq -k) = \Phi(k) - \big(1 - \Phi(k)\big) = 2\Phi(k) - 1.$$

Using the normal table values above, we obtain

$$\mathbf{P}(|X| \leq \sigma) = 0.6826, \qquad \mathbf{P}(|X| \leq 2\sigma) = 0.9544, \qquad \mathbf{P}(|X| \leq 3\sigma) = 0.9972,$$

where t is a standard normal random variable.