Math 30530 — Introduction to Probability

Quiz 2 – Wednesday September 25, 2013

Solutions

1. I shuffle a deck of 52 cards, and then deal them all out one after another. What is the probability that I see all four aces one after another with no cards in between? (If your answer involves factorials and/or binomial coefficients, you can leave them as is — no need for a complete numerical answer.)

   Solution: 1) number of arrangements of cards: 52!. number of arrangements with four aces together: think of four aces as "glued" into single card; there are 4! ways to do the gluing (one for each ordering of the our aces); this leaves a deck of 49 cards (48 ordinary, plus one "glued" card); there are 49! ways to order these cards; this gives a total of 4!49! good arrangements, for a probability of

   $$\frac{4!49!}{52!} = \frac{4.3.2}{52.51.50}.$$

   2) Probability of getting four aces, one after the other, right at the beginning:

   $$\left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right)\left(\frac{1}{49}\right).$$

   This is just the probability for the aces to appear right at the beginning. There are 49 places that the aces could possible begin, so the overall probability is

   $$49\left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right)\left(\frac{1}{49}\right) = \frac{4.3.2}{52.51.50}.$$

2. The math department wants to form an undergraduate committee with 8 people on in, exactly 4 of whom should be women. Among the faculty of 20, there are 7 are women. How many different possible committees are there? For this question, give your final answer as a number.

   Solution: First choose 4 women from among 7 ($\binom{7}{4}$ ways), then choose the remaining four committee members from among the 13 men ($\binom{13}{4}$ ways). Since this is a sequential experiment, the total number is the product of these two:

   $$\binom{7}{4}\binom{13}{4} = 25,025.$$

3. Show, either by an algebraic or a counting argument, that

   $$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

   Solution: 1) (Algebraic)

   $$k\binom{n}{k} = \frac{kn!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = \frac{n(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1}.$$

   2) (Counting) LHS counts number of ways of choosing a committee of size k, that has a chair, from a group of n people, by first choosing the committee ($\binom{n}{k}$ ways) and then choosing the chair from among the k that have been selected onto the committee (k ways).

RHS counts number of ways of choosing a committee of size k, that has a chair, from a group of n people, by first choosing the chair (n ways) and then choosing the rest of the committee from among the $n-1$ that were not selected to be chair ($\binom{n-1}{k-1}$ ways).

Since both sides count the same thing, they are equal.