Introduction to Probability, Fall 2013

Math 30530 Section 01

Homework 5 — due in class Friday, October 4

General information

At the top of the first page, write your name, the course number and the assignment number. If you use more than one page, you should **staple all your pages together**. The grader reserves the right to leave ungraded any assignment that is disorganized, untidy or incoherent.

Reading

- Sections 2.3 and 2.4

Problems

1. Chapter 2, problem 16
2. Chapter 2, problem 17
3. Chapter 2, problem 18
4. Chapter 2, problem 20 (For this problem, you may find the following trick useful: since $1 + x + x^2 + \ldots + x^k + \ldots = 1/(1-x)$, by differentiating both sides we get $1 + 2x + 3x^2 + 4x^3 + \ldots + kx^{k-1} + \ldots = 1/(1-x)^2$, and by differentiating again we get $2 + 6x + 12x^2 + 20x^3 + \ldots + k(k-1)x^{k-2} + \ldots = 2/(1-x)^3$)
5. Chapter 2, problem 22
6. Chapter 2, problem 23
7. Use the Taylor series of e^x to figure out the expected value of a Poisson random variable with parameter λ.
8. Let X be a negative Binomial random variable that counts the number of independent repetitions of a trial needed to obtain a total of m successes, when each individual trial has success probability p. Show that

$$\Pr(X > k) = \sum_{j=0}^{m-1} \binom{k}{j} p^j (1-p)^{k-j}.$$

(When $m = 1$ X becomes geometric with parameter p, and the equation reduces to $\Pr(X > k) = (1-p)^k$, which we discussed in class.)

9. In the Banach matchbox problem, suppose that we stop not when we first reach into a pocket and find an empty matchbox, but instead when we first reach into a pocket and realize that the match we are taking out is the last match in that pocket. Find the probability that at this moment the other pocket has exactly k matches. (Be sure to say what the possible values of k are).

10. Here is a winning strategy for Roulette: first, bet \$1 on red (so you win your dollar back, plus one more dollar, with probability 18/38, and you lose your dollar with probability 20/38). If red does come up, take your winnings and run. If not, then make additional bets on red for each of the next two spins of the wheel, and quit after that. Let X denote your net winnings at the end of this process.

    (a) Calculate $\Pr(X > 0)$ (your answer should be some number $> .5$, which shows that this is a winning strategy — you come out on top more times than you come out behind, on average)

    (b) Calculate the expected proportion of time in which you come out on top (with positive net winnings) while employing this strategy.

    (c) Calculate $E(X)$, the expected net winnings while employing this strategy.

11. Four busses carrying 148 students arrive at a school. The busses carry 40, 33, 25 and 50 students. A student is selected at random from among the 148. Let X be the number of students on the bus of the randomly selected student. A random driver (from among the four drivers) is selected at random. Let Y be the number of students on the bus of the randomly selected driver.

    (a) Without doing the calculations, which of $E(X)$, $E(Y)$ do you think might be bigger, and why?

    (b) Check your intuition by calculating $E(X)$ and $E(Y)$ and determining which is actually bigger.

12. On Monday afternoon you have \$1000 in your brokerage account, and nickel is selling for \$2 an ounce. You assess that on Tuesday morning, nickel will be worth either \$1 an ounce, or \$4 an ounce, each option equally likely. The only think you can do with your brokerage account is buy some nickel (not necessarily \$1000 worth) or keep money in the account.

    (a) Suppose your objective is to maximize the expected value of your portfolio (brokerage account balance plus value of nickel holdings) on late Tuesday morning. What should your Monday afternoon/Tuesday morning strategy be?

    (b) Suppose your objective is to maximize the expected amount (not value) of nickel you hold on late Tuesday morning. What should your Monday afternoon/Tuesday morning strategy be?

13. Suppose that X is a random variable that only takes on values $0, 1, 2, \ldots$. Show that

$$E(X) = \sum_{i=1}^{\infty} \Pr(X \geq i).$$

14. Let X be a binomial random variable with parameters n and p. Show that

$$E(1/(X+1)) = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$